Modeling time-varying conditional betas. A comparison of methods with application for REITs

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Abstract

Beta coefficients are the cornerstone of asset pricing theory in the CAPM and Multiple Factor models. This chapter proposes a review of different time series models used to estimate static and time-varying betas, and a comparison on real data. The analysis is performed on the U.S. and developed Europe REIT markets over the period 2009-2019 via a two-factor model. We evaluate the performance of the different techniques in terms of in-sample estimates as well as through an out-of-sample tracking exercise. Results show that dynamic models clearly outperform static models and that both the state space and Autoregressive Conditional Beta models outperform the other methods.

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1 Introduction

More than fifty years after its birth, Sharpe and Lintner's Capital Asset Pricing Model (CAPM) is still widely used by academics and practitioners to measure the performance of managed portfolios or to estimate the cost of equity for companies. A common practice within the financial industry consists in estimating linear time series models via an ordinary least square (OLS) regression, so that the slope coefficients (the betas) are assumed to be constant over the estimation period.

However, both theoretically and empirically, many studies have shown that betas may vary over time, meaning that the assumption of constancy is erroneous and potentially misleading. As a result, models that assume the constancy of parameters tend to be misspecified, which can lead to poor estimations, irrelevant forecasts and eventually bad financial decisions.

In order to model time-varying betas, three main alternatives to the simple OLS regression are typically used by academics and practitioners: using rolling-window OLS regressions, computing realized measures on sub-intervals of high-frequency data so as to obtain a realized beta and using exogenous interaction variables.

On top of the three alternatives listed above, many more methodologies have flourished over the years such as state space models with time-varying slope coefficients or Markov-switching models. More recently, two statistical approaches intended to capture the dynamic aspects of time series data in the case of multiple betas have been developed. Engle (2016), extending the work of Bollerslev et al. (1988), introduces a new model called the Dynamic Conditional Beta (DCB) model, offering a way to indirectly retrieve the time-varying slope coefficients of the independent variables via an estimate of the full conditional covariance matrix (using a multivariate GARCH model for instance). This approach is very intuitive and has the advantage of being easily implementable because MGARCH models are now available in many econometrics softwares. However, the approach also presents some major drawbacks. First, testing and imposing the constancy of the conditional betas is not so practical. Second, it is impossible to introduce exogenous variables in the model and to identify precisely which ones influence the evolution of the different betas since conditional betas are retrieved after a non-linear transformation of the elements that compose the estimated conditional covariance matrix instead of being modeled directly.

Darolles et al. (2018), extending the work of Pourahmadi (1999), take a different direction and offer a way to directly compute time-varying slope coefficients that depend on their lagged values and past shocks via a natural orthogonalization of the observed time series. Their model, called CHAR, which belongs to the class of MGARCH models, can also be used to obtain time-varying betas. The drawback of this approach is however that this method requires estimating the full multivariate system (like for the DCB model) even when one is interested in one equation only. To overcome this problem, building upon this method, Blasques et al. (2020) propose a new model, called Autoregressive Conditional Beta (ACB) model, that allows a direct modeling of the conditional betas. This model differs from the CHAR model due to the fact the dynamics of the conditional betas does not require the estimation of a system of equations but only a univariate model, with GARCH errors for instance.

After reviewing different ways to estimate both static and time-varying betas, we compare the performance of the most advanced conditional beta modeling techniques, that is to say the state space, DCB and ACB modeling techniques (with and without additional exogenous variables) to those of static betas (i.e. OLS and GARCH) in an empirical application focusing on the REIT market of the United States and developed Europe[1] using daily data over the period 2009-2019. To the best of our knowledge, we are the first to offer a comparison of these three conditional beta modeling techniques. In particular, we investigate the time variability of betas in a two-factor model, where $\beta_{B,t}$ and $\beta_{M,t}$ are respective measures of the sensitivity of the REIT index to changes in the bond market and the stock market. Results show that ACB models clearly outperform other competing models in-sample (when comparing the models on the basis of their log-likelihood) and that both the state space and ACB models outperform the other models out-of-sample (in a tracking exercise).

The remainder of the paper is organized as follows. Section 2 introduces the static beta model in a general framework and Section 3 presents different models used to estimate time-varying betas.[2] Section 4 describes the data and presents the results of the empirical application on REITs. Finally, Section 5 concludes.

2 Static betas

2.1 One-factor model

The fundamental equation common to most asset pricing models states that the price of a given asset i at time t, denoted $P_{i,t}$, must be equal to the expected discounted value of its payoff (for more details see, for instance, Cochrane, 2005, Ferson, 2003, and Smith and Wickens, 2002):

$$P_{i,t} = E_t[m_{t+1}(P_{i,t+1} + D_{i,t+1})], \tag{1}$$

where $D_{i,t+1}$ is the payment (interest or dividend) received at time $t+1$, and m_{t+1} is a strictly positive random variable used to discount the future payoffs, called the *stochastic discount factor* (SDF). $E_t(.) = E(.|\mathbf{\Omega}_t)$ denotes the time-t conditional expectation given the information set $\mathbf{\Omega}_t$.

Equation (1) assumes that the future payoff *and* discount factor[3] are stochastic: both are uncertain at date t and are contingent to future states of nature. It is worth noting that this equation holds for any investment horizon and type of asset (bond, share, option, real estate

[1]The term 'developed Europe' stands for the following countries: Austria, Belgium, Finland, France, Germany, Ireland, Italy, the Netherlands, Norway, Spain, Sweden, Switzerland and the United Kingdom.

[2]Three main approaches have been used in the literature to evaluate asset pricing models (see for instance Cochrane, 2005): time series modeling, cross-sectional regressions and calibration. This chapter only deals with time series models, excluding GMM estimates pioneered by Hansen (1982).

[3]The relationship between the discount factor m_{t+1} and the rate d_{t+1} at which future payoffs are discounted is: $m_{t+1} = \frac{1}{1+d_{t+1}}$. As a result, an increase in the discount factor m_{t+1} corresponds to a decrease in the stochastic discount rate d_{t+1}.

and so forth) and this without any specific assumption such as complete markets, financial market equilibrium, investor preference or distribution of asset returns.

Equation (1) can be expressed in terms of returns:

$$1 = E_t(m_{t+1} R_{i,t+1}), \tag{2}$$

where $R_{i,t+1} = \frac{P_{i,t+1}+D_{i,t+1}}{P_{i,t}}$ is the gross return of the asset at time $t+1$.

In the case of a risk-free asset whose gross return R_f is known with certainty and assumed to be constant, Equation (2) implies that the conditional expectation of the SDF is equal to the inverse of the risk-free rate:

$$E_t(m_{t+1}) = 1/R_f. \tag{3}$$

Equation (2) can be further developed as follows:

$$1 = E_t(m_{t+1})E_t(R_{i,t+1}) + cov_t(m_{t+1}, R_{i,t+1}), \tag{4}$$

where $cov_t(.)$ denotes the time-t expected conditional covariance given the information set $\mathbf{\Omega}_t$.

Substituting out Equation (3) into Equation (4) produces

$$1 = \frac{E_t(R_{i,t+1})}{R_f} + cov_t(m_{t+1}, R_{i,t+1}) \tag{5}$$

$$\Rightarrow E_t(R_{i,t+1}) - R_f = -R_f cov_t(m_{t+1}, R_{i,t+1}). \tag{6}$$

The risk premium $E_t(R_{i,t+1}) - R_f$ required to compensate risk-averse investors for holding risky assets only depends on the covariance of the payoffs with the discount factor, which is the only source of risk. This is due to the fact that investors have an incentive to pay more for assets with high payoffs in adverse conditions and the discount factor is precisely an index of adverse conditions (Cochrane and Culp, 2003).

The various asset pricing models differ in the way they model the discount factor. Most of them are based on the implications of consumer/investor intertemporal optimization models.[4] For instance, in the basic theoretical *Consumption CAPM model* (CCAPM) with time-additive preferences and power utility, the SDF corresponds to the inverse of the marginal rate of substitution between consumption today and consumption in the next period[5]

$$m_{t+1} = \frac{\beta U'(C_{t+1})}{U'(C_t)}, \tag{7}$$

where $U'(C_s)$ is the marginal utility of consumption at time s, denoted (C_s).

The combination of Equations (6) and (7) sheds light on the mechanism underlying the determination of asset prices. Assets whose returns covariate negatively with the future

[4] The first intertemporal CAPM model was developed by Merton (1973) in continuous time and led to a multi-factor model.

[5] Recall that the optimization of intertemporal utility in a deterministic model leads to the well-known Euler equation: $\frac{U'(C_t)}{\beta U'(C_{t+1})} = 1+r$, where r is the interest rate and the LHS is the marginal rate of substitution.

marginal utility of consumption are hedging assets because their returns tend to be high during times of market turbulence.[6] Consequently, the expected rate of return on these assets (and thus their risk premium) must be lower because investors have an incentive to buy them at a higher spot price.

A Taylor's series expansion of Equation (7) leads to a linear relationship between m_{t+1} and the growth of consumption (Δc_{t+1}):[7]

$$m_{t+1} \approx \beta - \beta\kappa_t \Delta c_{t+1}, \tag{8}$$

where $\kappa_t = -\dfrac{C_t U"(C_t)}{U'(C_t)}$ is the consumer/investor's subjective relative risk aversion coefficient.

Interesting results can be obtained by focusing on wealth rather than on consumption. Assume that wealth is held through the market portfolio[8], i.e. the portfolio of all market assets weighted according to their relative value. Let $R_{M,t+1} = \dfrac{W_{t+1}}{W_t}$ be the gross return of the market portfolio and $r_{M,t+1} = R_{M,t+1} - 1$ the net return. The investor's optimization leads to[9]

$$m_{t+1} \approx 1 + \frac{W_t U"(W_{t+1})}{U'(W_t)} r_{M,t+1} \tag{9}$$

$$\Rightarrow m_{t+1} \approx (1-\gamma_t) - \gamma_t R_{M,t+1}, \tag{10}$$

where $U'(W_s)$ is the marginal utility of W_s (the wealth at time s), and $\gamma_t = -\dfrac{W_t U"(W_{t+1})}{U'(W_t)}$ is the investor's subjective relative risk aversion coefficient.[10]

Dividing both terms of Equation (6) by $var_t(R_{i,t+1}|\mathbf{\Omega}_t)$, we ge

$$\frac{E_t(R_{i,t+1}) - R_f}{var_t(R_{i,t+1})} = -R_f \frac{cov_t(m_{t+1}, R_{i,t+1})}{var_t(R_{i,t+1})}, \tag{11}$$

where $var_t(.) = var_t(.|\mathbf{\Omega}_t)$ denotes the time-t expected conditional variance.

Using Equation (10) and assuming for simplicity that $Cov_t(\gamma_t, R_{i,t+1}) = 0$, we obtain for

[6] Since the marginal utility of consumption decreases with the consumption level, an increase in the future marginal utility corresponds to an unfavorable state of nature.

[7] See Smith and Wickens (2002).

[8] The market portfolio may in theory include financial assets, consumer durables, real estate and human capital (Roll, 1977; Fama and French, 2004). Consequently, the gross market returns $R_{M,t+1}$ proxied by the return on an equity index is thus a rather narrow measure.

[9] See Harvey and Siddique (2000) as well as Phelan and Toda (2015). More to the point, Harvey and Siddique (2000) suggest a second order approximation of Equation (9) which leads to a non-linear (quadratic) relation between m_{t+1} and $R_{M,t+1}$ in Equation (10).

[10] As stated by Cochrane (2005), p. 464, γ_t in Equation (10) represents aversion to bets on wealth while κ_t in Equation (8) represents aversion to bets on consumption. γ_t is thus a more intuitive measure of risk aversion.

$i = M$:

$$\frac{E_t(R_{M,t+1}) - R_f}{var_t(R_{M,t+1})} = -R_f \frac{cov_t(m_{t+1}, R_{M,t+1})}{var_t(R_{M,t+1})} \tag{12}$$

$$\Rightarrow \frac{E_t(R_{M,t+1}) - R_f}{var_t(R_{M,t+1})} = \gamma_t R_f. \tag{13}$$

The result given in Equation (13) is fairly usual in conventional portfolio theories. Indeed, the risk premium required by an investor to hold the market portfolio is equal to his/her risk aversion multiplied by the conditional variance of the market portfolio (i.e. the risk): $E_t(R_{M,t+1}) - R_f \ = R_f \gamma_t var_t(R_{M,t+1})$.[11]

Substituting out Equations (10) and (13) into Equation (6), the pricing of asset i gives

$$
\begin{aligned}
E_t(R_{i,t+1}) - R_f &= -R_f cov_t(m_{t+1}, R_{i,t+1}) && (14)\\
&= \gamma_t R_f cov_t(R_{M,t+1}, R_{i,t+1}) && (15)\\
&= \frac{E_t(R_{M,t+1}) - R_f}{var_t(R_{M,t+1})} cov_t(R_{M,t+1}, R_{i,t+1}) && (16)\\
&= \beta_{i,t} \left[E_t(R_{M,t+1}) - R_f \right], && (17)
\end{aligned}
$$

where

$$\beta_{i,t} = \frac{cov_t(R_{M,t+1}, R_{i,t+1})}{var_t(R_{M,t+1})}. \tag{18}$$

Equation (17) can be expressed more conveniently in the following form:

$$E_t\left(\widetilde{r}_{i,t+1}\right) = \beta_{i,t} E_t\left(\widetilde{r}_{M,t+1}\right), \tag{19}$$

where $E_t\left(\widetilde{r}_{i,t+1}\right) = E_t(R_{i,t+1}) - R_f$, $E_t\left(\widetilde{r}_{M,t+1}\right) = E_t(R_{M,t+1}) - R_f$, i.e. respectively the conditional expectation of the net excess return of asset i and of the market.[12]

Equation (19) is the *Conditional CAPM model*, i.e. a conditional version of the theoretical CAPM proposed by Sharpe (1964) and Lintner (1965). If $\beta_{i,t}$ is constant and conditional information plays no role in determining excess returns, then Equation (19) becomes[13]

$$E(\widetilde{r}_i) = \beta E(\widetilde{r}_M), \tag{20}$$

[11] Moreover, Equation (13) states that if the risk is measured by the conditional variance, the market price of risk (LHS) is equal to the investor's subjective relative risk aversion (discounted by the risk-free rate).

[12] Recall that the net return of an asset i is linked to the gross return by the definition $r_{i,t+1} = R_{i,t+1} - 1$.

[13] The basic Sharpe and Lintner's unconditional CAPM rests on more restrictive assumptions than the CCAPM given in Equation (17), since it results from the maximization of a single period mean-variance criterion. As stated by Merton (1973), the single-period utility function only coincides with intertemporal maximization when preferences and future investment opportunity sets are not state dependent.

As a consequence, the validity of the conditional CAPM does note imply the validity of the unconditional CAPM. As noted by Wang (1996), the unconditional CAPM can differ from the unconditional expectations of Equation (17) if $Cov(\beta_{i,t}, R_{M,t+1}) \neq 0$, and thus the unconditional beta may differ from the expected conditional beta (Lewellen and Nagel, 2006). However, if $\beta_{i,t}$ is a deterministic constant ($\beta_{i,t} = \beta, \forall t$) then the unconditional CAPM and the unconditional expectation of the conditional CAPM are equivalent.

where $E(\widetilde{r}_i)$ and $E(\widetilde{r}_M)$ are the unconditional expectations of net returns, and β (we omit the index i to simplify the notation) is the unconditional market beta of asset i.

The CAPM is the most widely studied asset valuation model and is used to estimate the required rate of return on an asset given a certain level of systematic risk (or market risk), expressed as the market beta β. While investors are facing two types of risks when investing, idiosyncratic and systematic risks, only systematic risk is priced since idiosyncratic risk can totally be offset through diversification. In the theoretical CAPM, market risk is the only source of systematic risk.

In this framework, the most basic way to estimate the beta of an asset is to run the simple bivariate OLS regression over the whole sample:

$$\widetilde{r}_{i,t} = \alpha + \beta \widetilde{r}_{M,t} + \varepsilon_t \tag{21}$$

$$\varepsilon_t \overset{i.i.d.}{\sim} D(0, \sigma^2), \tag{22}$$

where ε_t captures the idiosyncratic risk of asset i and follows a D distribution (e.g. Gaussian) with mean 0 and constant variance σ^2.

In this case, and from the market model regression, α is expected to be zero. The OLS estimate of β, is obtained as

$$\hat{\beta}^{OLS} = \frac{cov(\widetilde{r}_i, \widetilde{r}_M)}{var(\widetilde{r}_M)}, \tag{23}$$

where $Cov(\widetilde{r}_i, \widetilde{r}_M)$ is the unconditional covariance between the asset's excess return and the market excess return, and $Var(\widetilde{r}_M)$ is the unconditional variance of the market excess return.

One way to relax the restrictive $i.i.d.$ hypothesis on the residuals is to use a univariate GARCH model. The standard GARCH (1,1) model offers the advantage of accounting for two features which are commonly observed in financial time series data, i.e. leptokurtosis (distribution's excess peakedness and fat tails) and volatility clustering (tendency for low/high volatility to persist and appear in bunches). In this case, the residuals and the conditional variance can be specified as

$$\varepsilon_t = \sigma_t z_t, \quad z_t \overset{i.i.d.}{\sim} D(0,1), \tag{24}$$

$$\sigma_t^2 = \lambda_0 + \lambda_1 \sigma_{t-1}^2 + \lambda_2 \varepsilon_{t-1}^2, \tag{25}$$

where z_t is an $i.i.d.$ random variable with mean 0 and unit variance while the conditional variance at time t, denoted σ_t^2, depends on both the lagged squared error term at time $t-1$ and the lagged variance term at time $t-1$.

2.2 Multiple betas

The Consumption CAPM and conditional CAPM are basically one-factor asset pricing models. Indeed, Equations (8) and (10) belong to the more general class of *linear pricing kernel*

$$m_{t+1} \approx a_{0,t} + a_{1,t} F_{1,t+1} + ... + a_{N,t} F_{N,t+1} \tag{26}$$

in which $F_{1,t+1}, ..., F_{N,t+1}$ are variables or factors that are good proxies for growth of marginal utility.

Many authors have considered the case where the stochastic discount factor can be represented as a linear function of N factors of the form given by Equation (26). For instance, in Equation (10), which leads to a conditional CAPM, the return on the market portfolio is proxied by the return on a stock market index $R_{M,t+1}$, but this assumption is criticized by Roll (1977), who argues that this approximation neglects the human capital component in total wealth. Wang (1996) and other authors suggest that the growth rate of labor income can be a good proxy for the return on human capital. Under this hypothesis, one can consider an SDF with two factors: the return of a stock market index $R_{M,t+1}$ and the growth rate of labor income Δy_{t+1}: $m_{t+1} \approx a_{0,t} + a_{1,t} R_{M,t+1} + a_{2,t} \Delta y_{t+1}$.

Considering Equation (26), Ferson and Jagannathan (1996) show that in the case where the factors $F_{1,t+1}, ..., F_{N,t+1}$ are traded assets[14] and if the coefficients are defined as follows[15]

$$\begin{aligned} a_{j,t} &= -\frac{E_t\left(\widetilde{f}_{j,t+1}\right)}{R_f var_t(F_{j,t+1})}, \quad j = 1, ..., N && (27) \\ a_{0,t} &= \frac{1}{R_f} - \sum_{j=1}^{N} a_{j,t}, && (28) \end{aligned}$$

where $E_t\left(\widetilde{f}_{j,t+1}\right) = E_t(F_{j,t+1}) - R_f$ denotes the (conditional) expected risk premium of factor j, then we obtain a multi-factor representation of the conditional CAPM:

$$E_t(\widetilde{r}_{i,t+1}) = \sum_{n=1}^{N} \beta_{n,t} E_t(\widetilde{f}_{n,t}). \tag{29}$$

As we have done previously, if all $\beta_{n,t}$'s are constant and conditional information plays no role in determining excess returns, we obtain an unconditional factor model which looks like the equation used in the so-called Arbitrage Pricing Theory (APT) initiated by Ross (1976), i.e.,

$$\widetilde{r}_{i,t} = \alpha + \sum_{n=1}^{N} \beta_n \widetilde{f}_{n,t} + \varepsilon_t. \tag{30}$$

Equation (30) can be written in a more compact and standard form as follows:

$$y_t = \mathbf{x}_t' \boldsymbol{\beta} + \varepsilon_t, \tag{31}$$

where $y_t = \widetilde{r}_{i,t}$, $\mathbf{x}_t = [1, \widetilde{f}_{1,t}, ..., \widetilde{f}_{N,t}]'$ and $\boldsymbol{\beta} = (\alpha, \beta_1, ..., \beta_N)'$.

[14] If one of the factors is not a traded asset return, then its expected risk premium is estimated by the conditional expectation of the excess return of the factor mimicking portfolio, i.e. a portfolio whose returns can be used instead of the factor itself. See for e.g. Ferson (2003).

[15] If the jth factor is the market return, then $a_{j,t} = -\gamma_t$, i.e. the negative of the coefficient of time-varying relative risk aversion given in Equation (13).

The OLS estimator of the parameter vector $\boldsymbol{\beta} = (\alpha, \beta_1, \dots, \beta_N)'$ is

$$(\hat{\alpha}, \hat{\beta}_1, \dots, \hat{\beta}_N)' = (\mathbf{x}'\mathbf{x})^{-1}\mathbf{x}'\mathbf{y}. \tag{32}$$

In the specific case where the covariance matrix of the regressors is diagonal (that is, if the different factors are mutually uncorrelated) this formula leads to a simple generalization of Equation (23) since

$$\hat{\beta}_n^{OLS} = \frac{cov(\widetilde{f}_n, \widetilde{r}_i)}{var(\widetilde{f}_n)}, \forall n = 1, \dots, N. \tag{33}$$

According to the APT, the one-factor CAPM is not appropriate in a world with multiple risk factors represented by microeconomic or macroeconomic variables. Among possible risk factors, one can mention inflation, the spread between short-term and long-term bonds, industrial production growth or default risks (Brooks, 2014). Many pricing models have been developed with additional risk factors such as the size and value risk factors (Fama and French, 1993), the momentum risk factor (Carhart, 1997) or the profitability and investment risk factors (Fama and French, 2015).

However, multi-factor pricing models have been criticized for poor out-of-sample performance and for data snooping (see for instance Andersen et al., 2003, Harvey et al., 2015, and Linnainmaa and Roberts, 2018).

According to Harvey et al. (2015), more than 300 factors have been presented in the literature as important and significant in explaining the cross-sectional variation of stock returns. Many of these factors are however difficult to interpret from an economic perspective.

3 Time-varying betas

There is a large consensus in the literature about the fact that betas are actually time-varying. Such evidence has been pointed out by Fama and MacBeth (1973), Fabozzi and Francis (1977), Alexander and Chervany (1980), Sunder (1980), Ohlson and Rosenberg (1982), DeJong and Collins (1985), Fisher and Kamin (1985), Brooks et al. (1992), Brooks et al. (1994) among others. The aim of this section is therefore to review various methods used to estimate time-varying betas (and potentially time-varying alphas) in a conditional model close to Equation (29):

$$\widetilde{r}_{i,t} = \alpha_t + \sum_{n=1}^{N} \beta_{n,t} \widetilde{f}_{n,t} + \varepsilon_t, \tag{34}$$

or equivalently (31):

$$y_t = \mathbf{x}_t' \boldsymbol{\beta}_t + \varepsilon_t. \tag{35}$$

3.1 Rolling betas

The first and simplest way to estimate time-varying betas is to estimate beta over moving sub-periods using a simple rolling-window OLS regression as proposed by Fama and MacBeth

(1973) (see Van Nieuwerburgh, 2019 and Zhou, 2013 for applications in the case of the REIT beta).

Let us assume we have historical data on a period spanning from t_0 to t_T. This method consists in selecting a window of h observations (for instance 120 daily returns) and then estimating Equation (30) by OLS over the first h observations, that is from t_0 to t_{0+h}, so as to obtain $(\hat{\alpha}_h, \hat{\beta}_{1,h}, \ldots, \hat{\beta}_{N,h})'$. Then, the window is rolled one step forward by adding one new observation and dropping the most distant one. So, $(\hat{\alpha}_{h+1}, \hat{\beta}_{1,h+1}, \ldots, \hat{\beta}_{N,h+1})'$ are obtained in the same way over the period t_{0+1} to t_{0+h+1} and the sequence is reiterated until the end of the sample period.

This method can be considered as a quick and dirty time-varying regression model. When compared to the usual OLS regression model, the rolling-window OLS regression model offers the advantage of taking into account time variations in the alpha and the betas. However, the coefficients measured with this method only vary very slowly by construction due to the fact only one period is dropped and another is added between two successive estimates, which can lead to inaccurate estimates. The results also depend heavily on the size of the chosen window.[16]

3.2 Realized betas

An alternative estimation method of time-varying betas is to compute both the realized variance and realized covariance from intraday[17] data so as to estimate the beta using so called *realized measures*. The realized variance and covariance being computed from intraday data, they are much more accurate than standard measures (Hansen et al., 2014). This approach, based on the works of Andersen et al. (2003) and Barndorff-Nielsen and Shephard (2004)[18], offers a more accurate way of analyzing the dynamic behavior of beta than the rolling beta methodology (Patton and Verardo, 2012). In the context of a single factor model, the realized market beta, denoted β^R, is defined as

$$\beta_t^R = \frac{cov^R(\widetilde{r}_i, \widetilde{r}_M)_t}{var^R(\widetilde{r}_M)_t} = \frac{\Sigma_{k=1}^{(s)} \widetilde{r}(t)_{i,k} \widetilde{r}(t)_{M,k}}{\Sigma_{k=1}^{(s)} \widetilde{r}(t)_{M,k}}, \tag{36}$$

where $cov^R(\widetilde{r}_i, \widetilde{r}_M)$ is the realized covariance between the asset's excess return and the market excess return, and $var^R(\widetilde{r}_M)$ is the realized variance of the market excess return. $\widetilde{r}(t)_{i,k}$ is the excess return on asset i during the kth intraday period on day t and s is the total number of intraday periods.[19] As a consequence, the realized beta is the ratio of an asset's

[16]Some refinements can be made, for example, by introducing a weighting scheme giving less weight to observations from more distant periods (see for instance Nieto et al., 2014).

[17]Intraday data are used to estimate daily beta, variances or covariances, assuming they are fairly stable during the day. It is of course possible to use, for instance, daily data to compute beta, variances or covariances over a month if we assume that these parameters are fairly stable during each month. See for instance Andersen et al. (2006) and Lewellen and Nagel (2006).

[18]This methodology assumes an absence of jumps. In the case of jumps, see Todorov and Bollerslev (2010).

[19]For more details, see Patton and Verardo (2012).

sample covariance with the market to the sample variance of the market over several intraday periods.[20]

This method resembles the rolling beta approach but it relies on non-overlapping windows (of one day or one month for instance) and requires data sampled at a higher frequency within each window (e.g. 5-minute returns). The main limit of this methodology is therefore the need for a sufficient number of intraday data in order to be able to compute the realized variance and covariance. See Boudt et al. (2017) for an extension to multi-factor betas.

3.3 Time-varying betas with interaction variables

Rolling OLS and 'realized betas' do not rely on a parametric model to specificy the dynamics of the time-varying betas. Shanken (1990) and Schadt (1996) introduce the hypothesis that the dynamics of the betas depends on a set of exogenous variables. This leads to an alternative way of modeling conditional betas, via the use of interaction variables (see also Gagliardini et al., 2016).

Indeed, interaction variables can be used to introduce dynamics in $\beta_{n,t}$ in the following way:

$$\beta_{n,t} = \beta_n + \sum_{k=1}^{K} \theta_{n,k} Z_{k,t-1}, \tag{37}$$

where each $Z_{k,t-1}$ variable ($k = 1, \ldots, K$) is an observable state variable, predetermined at the end of period $t-1$, and assumed to drive the dynamics of the beta of the n'th factor, and where $\theta_{n,k}$ is its associated coefficient. The above model can be rewritten as

$$\widetilde{r}_{i,t} = \alpha + \beta \widetilde{r}_{M,t} + \sum_{k=1}^{K} \theta_k Z_{k,t-1} \widetilde{r}_{M,t} + \varepsilon_t$$

and therefore corresponds to a multiple linear regression model with k interaction variables when the conditional variance is constant or a GARCH model with k interaction variables in the conditional mean when the conditional variance is assumed to follow a GARCH dynamics.

The advantage of this method is that the model can easily be estimated. However, this method requires the selection of suitable variables supposed to drive the dynamics of the betas. In addition, they must be able to generate a certain persistence that can be observed in the dynamic behavior of the betas.

Many studies have used interaction variables. It seems worth mentioning Schwert and Seguin (1990), in which betas are assumed to vary with the level of aggregate market volatility. The authors estimate the conditional market beta in the following way:

$$\beta_{i,t} = \beta_i + \theta_i \left(\frac{1}{\hat{\sigma}^2_{M,t}} \right), \tag{38}$$

[20] It is also possible, as in Lewellen and Nagel (2006), to directly estimate Equation (30) for each intraday (or intra-monthly, intra-quarterly,...) period.

where β_i and θ_i are constant parameters and $\hat{\sigma}^2_{M,t}$ is the time-varying volatility of the aggregate stock market. The model is mainly used to account for time-variation in stock betas or to study the relationship between firm size and time-varying betas (see for example Reyes, 1999).

3.4 Indirect dynamic conditional betas

As we have seen, the theoretical conditional CAPM expresses the conditional market beta as follows:

$$\beta_{i,t} = \frac{cov_t(R_{M,t+1}, R_{i,t+1})}{var_t(R_{M,t+1})}. \tag{39}$$

This expression opens up the possibility of obtaining time-varying betas from the estimation of conditional variances and covariances obtained for instance by a multivariate GARCH model (see for instance Bali, 2010 for an application and Bauwens et al., 2006 for a survey on MGARCH models). In this case, the model imposes a minimal structure on the time-varying process, apart from the modeling of conditional variances and covariances in an autoregressive form. However, Equation (39) does not apply in the multi-factor model when the factors are correlated.

Engle (2016) recently extended the multivariate GARCH approach to the case of a multi-factor model. Following Engle's (2016) methodology, the conditional betas are inferred from an estimate of the conditional covariance matrix $\mathbf{\Sigma}_t$ of $(\mathbf{x}_t, y_t)'$.

For ease of exposition we assume in this section that $\mathbf{x}$ and y have been centered so that $\mathbf{x}$ does not contain a vector of ones (corresponding to α) and therefore one does not need to estimate the intercept in (31).

In order to obtain the coefficients of the multivariate regression of y_t (asset returns) on $\mathbf{x}_t$ (factors), Engle (2016) assumes that $(\mathbf{x}_t, y_t)'$ follows an $(N+1)$-dimensional normal distribution (conditional on the information set at time $t-1$, denoted $\mathcal{F}_{t-1}$), i.e.

$$\begin{pmatrix} \mathbf{x}_t \\ y_t \end{pmatrix} | \mathcal{F}_{t-1} \sim N\left(\begin{pmatrix} \mathbf{0}_{m-1} \\ 0 \end{pmatrix}, \mathbf{\Sigma}_t \equiv \begin{pmatrix} \mathbf{\Sigma}_{\mathbf{xx},t} & \mathbf{\Sigma}_{\mathbf{x}y,t} \\ \mathbf{\Sigma}_{y\mathbf{x},t} & \mathbf{\Sigma}_{yy,t} \end{pmatrix} \right),$$

where subscripts embody natural partitions.

In order to derive an estimate of the conditional betas, Engle (2016) relies on the fact that the conditional distribution of y_t on $\mathbf{x}_t$ is

$$y_t|\mathbf{x}_t \sim N\left(\mathbf{\Sigma}_{y\mathbf{x},t}\mathbf{\Sigma}^{-1}_{\mathbf{xx},t}\mathbf{x}_t, \mathbf{\Sigma}_{yy,t} - \mathbf{\Sigma}_{y\mathbf{x},t}\mathbf{\Sigma}^{-1}_{\mathbf{xx},t}\mathbf{\Sigma}_{\mathbf{x}y,t}\right). \tag{40}$$

In more details, estimates of the time-varying coefficients inferred from the regression of y_t on $\mathbf{x}_t$ can be retrieved from $\mathbf{\Sigma}_t$ as follows:

$$\widehat{\boldsymbol{\beta}}^{DCB}_t \equiv (\hat{\beta}^{DCB}_{1,t}, \ldots, \hat{\beta}^{DCB}_{N,t})' = \mathbf{\Sigma}^{-1}_{\mathbf{xx},t}\mathbf{\Sigma}_{\mathbf{x}y,t}. \tag{41}$$

When there is only one regressor, estimates of the time-varying coefficients inferred from the regression of y_t on x_t can simply be retrieved from $\Sigma_{xy,t}/\Sigma_{xx,t}$, i.e. the conditional covariance between y_t and x_t divided by the conditional variance of x_t. While any MGARCH model can be used to estimate $\mathbf{\Sigma}_t$, Engle (2016) uses a Dynamic Conditional Correlation (DCC) GARCH model on $(\mathbf{x}_t, y_t)'$, which relies on the following decomposition of $\mathbf{\Sigma}_t$:

$$\mathbf{\Sigma}_t = \mathbf{D}_t \mathbf{R}_t \mathbf{D}_t, \tag{42}$$

where $\mathbf{D}_t$ is a diagonal matrix containing the conditional volatilities (typically modeled with $N+1$ independent univariate GARCH models) while $\mathbf{R}_t$ is a conditional correlation matrix (usually obtained as a transformation of a scalar BEKK specification on the devolatilized series). Note that the Constant Conditional Correlation (CCC) model proposed by Bollerslev et al. (1990) is obtained when $\mathbf{R}_t = \mathbf{R}$.

Based on the DCC-GARCH model, the DCB model presents several drawbacks, which we summarize here. First, the stationarity and ergodicity conditions of the DCC are not well known. Second, the model incorporates complicated constraints associated with correlation matrices. Third, the asymptotic properties of the QMLE (Quasi Maximum Likelihood Estimator) are unknown. And fourth, the effects of the DCC parameters on β_t are nearly impossible to interpret.

Note that out-of-sample forecasts of the betas can be obtained by adapting Equation (41) to out-of-sample forecasts of the conditional covariance matrix.

3.5 Direct dynamic conditional betas

In this section, we present two competing models that directly specify the dynamics of the conditional betas: the first model belongs to the general class of data-driven models while the second model belongs to the class of observation-driven models.

3.5.1 State-space models

Adrian and Franzoni (2009) suggest a stylized model based on the conditional CAPM, in which beta changes over time and the investor's expectation of beta results from a learning process. This learning process is modeled via a Kalman process in which beta is treated as a latent variable. Adrian and Franzoni (2009) thus provide a theoretical foundation for the estimation of unobserved time-varying betas by state-space modeling (see Choudhry and Wu, 2008, Cisse et al., 2019, Faff et al., 2000, Huang, 2009, Mergner, 2008, and Nieto et al., 2014 among others).

A useful state space representation of the multi-factor model is given by the following system of equations:

$$y_t = \mathbf{x}_t' \boldsymbol{\beta}_t + \varepsilon_t \tag{43}$$

$$\boldsymbol{\beta}_t - \overline{\boldsymbol{\beta}} = \mathbf{\Phi}(\boldsymbol{\beta}_{t-1} - \overline{\boldsymbol{\beta}}) + \mathbf{u}_t, \tag{44}$$

where $y_t = \widetilde{r}_{i,t}$, $\mathbf{x}_t = [1, \widetilde{f}_{1,t}, ..., \widetilde{f}_{N,t}]'$, $\boldsymbol{\beta}_t = (\alpha_t, \beta_{1,t}, ..., \beta_{N,t})'$, $\mathbf{\Phi}$ is a $((N+1) \times (N+1))$ transition matrix which can be assumed to be diagonal: $\mathbf{\Phi} = diag(\Phi_0, \Phi_1, ..., \Phi_N)$.

The $N+2$ residuals are assumed to be conditionally $i.i.d$ and mutually independent, i.e.

$$\begin{pmatrix} \mathbf{u}_t \\ \varepsilon_t \end{pmatrix} | \mathbf{Y}_{t-1}, \mathbf{X}_t \overset{i.i.d}{\sim} N\left(\begin{pmatrix} \mathbf{0} \\ 0 \end{pmatrix}, \begin{pmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \sigma^2 \end{pmatrix} \right), \tag{45}$$

where $\mathbf{Q} = diag(\sigma^2_{u0}, \sigma^2_{u1}, ..., \sigma^2_{uN})$, and we use the notations $\mathbf{Y}_t = \{y_t, y_{t-1}, ...y_1\}$ and $\mathbf{X}_t = \{\mathbf{x}'_t, \mathbf{x}'_{t-1}, ...\mathbf{x}'_1\}$ to denote observations available at time t.[21]

Equation (44) encompasses several specifications (see Chapter 2 of Moryson, 1998, for more details).

If $\mathbf{\Phi} = \mathbf{I}$, each conditional beta follows a *random walk* process, i.e.

$$\beta_{n,t} = \beta_{n,t-1} + u_{n,t}, \qquad n = 1, \ldots, N. \tag{46}$$

In this case, the h-step-ahead out-of-sample forecast of the n-th conditional beta at the end of the estimation period is given by $\beta_{n,T+h} = \beta_{n,T}$, $\forall h > 0$.

If $\mathbf{\Phi} = \mathbf{0}$, we obtain the *random coefficient* model

$$\beta_{n,t} = \overline{\beta}_n + u_{n,t}, \qquad n = 1, \ldots, N, \tag{47}$$

where the deviation of $\beta_{n,t}$ from its unconditional mean $\overline{\beta}_n$ is caused solely by the noise $u_{n,t}$, and the h-step-ahead out-of-sample forecasts is $\beta_{n,T+h} = \overline{\beta}_n$, $\forall h > 0$.

Finally, Equation (44) corresponds to the *mean reverting* model[22]

$$\beta_{n,t} = \overline{\beta}_n + \Phi_n(\beta_{n,t-1} - \overline{\beta}_n) + u_{n,t}. \tag{48}$$

The stochastic process $\beta_{n,t}$ reverts to its unconditional mean $\overline{\beta}_n$ after a shock and the parameter Φ_n controls the speed of reversion to the mean. The h-step-ahead out-of-sample forecast is given by $\beta_{n,T+h} - \overline{\beta}_n = \Phi_n^h(\beta_{n,T} - \overline{\beta}_n)$.

Let us write Equation (44) in a more compact form as

$$\boldsymbol{\beta}_t = \boldsymbol{\mu} + \mathbf{\Phi}\boldsymbol{\beta}_{t-1} + \mathbf{u}_t, \tag{49}$$

where $\boldsymbol{\mu} = (\alpha(1 - \Phi_0), \beta_1(1 - \Phi_1), ..., \beta_N(1 - \Phi_N))'$.

Estimation of the model's parameters can be achieved by the Kalman filter, an iterative algorithm producing at each time t an estimator of $\boldsymbol{\beta}_t$ denoted $\widehat{\boldsymbol{\beta}}_{t|t-1}$ based on the information up to time $t-1$.

Given an estimate $\widehat{\boldsymbol{\beta}}_{t|t-1}$, the measurement Equation (43) can be written as

$$y_t = \mathbf{x}'_t\widehat{\boldsymbol{\beta}}_{t|t-1} + \mathbf{x}'_t(\boldsymbol{\beta}_t - \widehat{\boldsymbol{\beta}}_{t|t-1}) + \varepsilon_t. \tag{50}$$

The one period-ahead conditional forecast is thus $E[y_t|\mathbf{Y}_{t-1}, \mathbf{X}_t] = y_{t|t-1} = \mathbf{x}'_t\widehat{\boldsymbol{\beta}}_{t|t-1}$ and the prediction error

$$\begin{aligned} \eta_{t|t-1} &= y_t - y_{t|t-1} \\ &= y_t - \mathbf{x}'_t\widehat{\boldsymbol{\beta}}_{t|t-1}. \end{aligned} \tag{51}$$

[21]Several extensions can be accounted for, such as time varying variances/covariances of the error terms.

[22]Since we assume $\mathbf{\Phi} = diag(\Phi_0, \Phi_1, ..., \Phi_N)$, if all $(\Phi_0, \Phi_1, ..., \Phi_N)$ are inside of the unit circle, then the vector $\overline{\boldsymbol{\beta}}$ corresponds to the average value of $\boldsymbol{\beta}_{t+1}$.

From Equation (50), the MSE of the prediction error can be easily computed as

$$f_{t|t-1} = E[(\eta_{t|t-1})^2|\mathbf{Y}_{t-1}, \mathbf{X}_t] = \mathbf{x}_t'\mathbf{P}_{t|t-1}\mathbf{x}_t + \sigma^2, \tag{52}$$

where $\mathbf{P}_{t|t-1} = E[(\boldsymbol{\beta}_t - \widehat{\boldsymbol{\beta}}_{t|t-1})(\boldsymbol{\beta}_t - \widehat{\boldsymbol{\beta}}_{t|t-1})'|\mathbf{Y}_{t-1}, \mathbf{X}_t]$ is the covariance matrix of $\boldsymbol{\beta}_t$ conditional on the information up to $t-1$. We can see from Equation (52) that the MSE of the prediction error consists of two parts: the uncertainty associated with $\widehat{\boldsymbol{\beta}}_{t|t-1}$ and the variance of ε_t.

Under the gaussianity assumption, the sample log-likelihood is (Hamilton, 1994, Durbin and Koopman, 2001):

$$\begin{aligned}\sum_{t=1}^{T} \log f(y_t|\mathbf{Y}_{t-1}, \mathbf{X}_t) = -\left(\frac{T}{2}\right)\log(2\pi) - \left(\frac{1}{2}\right)\sum_{t=1}^{T}\log(f_{t|t-1}) \\ -\sum_{t=1}^{T}\frac{\left(\eta_{t|t-1}\right)^2}{f_{t|t-1}},\end{aligned} \tag{53}$$

where $\eta_{t|t-1}$ and $f_{t|t-1}$ are given respectively in Equations (51) and (52).

With initial condition $\widehat{\boldsymbol{\beta}}_{0|0}$ and $\mathbf{P}_{0|0}$, the conditional covariance matrix $\mathbf{P}_{t|t-1}$ and the conditional vector $\widehat{\boldsymbol{\beta}}_{t+1|t}$ are recursively computed according to the following prediction equations:

- One-step ahead forecast

$$\widehat{\boldsymbol{\beta}}_{t|t-1} = \boldsymbol{\mu} + \boldsymbol{\Phi}\widehat{\boldsymbol{\beta}}_{t-1|t-1} \tag{54}$$

$$\mathbf{P}_{t|t-1} = \boldsymbol{\Phi}\mathbf{P}_{t-1|t-1}\boldsymbol{\Phi}' + \mathbf{Q} \tag{55}$$

  as well as Equations (51) and (52).

- Kalman gain

$$\mathbf{K}_t = \mathbf{P}_{t|t-1}\mathbf{x}_t' f_{t|t-1}^{-1}. \tag{56}$$

- Measurement update

$$\widehat{\boldsymbol{\beta}}_{t|t} = \widehat{\boldsymbol{\beta}}_{t|t-1} + \mathbf{K}_t\eta_{t|t-1} \tag{57}$$

$$\mathbf{P}_{t|t} = (\mathbf{I} - \mathbf{K}_t\mathbf{x}_t')\mathbf{P}_{t|t-1}. \tag{58}$$

The Kalman gain $\mathbf{K}_t$ is used to update $\widehat{\boldsymbol{\beta}}_{t|t}$ from $\widehat{\boldsymbol{\beta}}_{t|t-1}$. It determines the relative weighting of new information (given by the prediction error $\eta_{t|t-1}$) versus the current state estimate $\widehat{\boldsymbol{\beta}}_{t|t-1}$.

For given parameters of the model, the recursive Equations (54)-(58) provide the prediction error $\eta_{t|t-1}$ and its variance $f_{t|t-1}$. The unknown parameters are estimated through the maximization of the log-likelihood (53) with respect to these parameters, using $\eta_{t|t-1}$ and $f_{t|t-1}$ as inputs.

Finally, the smoothed estimates are obtained by iterating backward for $t = T-1, T-2, ..., 1$ the following equations:

$$\widehat{\boldsymbol{\beta}}_{t|T} = \widehat{\boldsymbol{\beta}}_{t|t} + \mathbf{P}_{t|t}\boldsymbol{\Phi}'\mathbf{P}^{-1}_{t+1|t}\left(\widehat{\boldsymbol{\beta}}_{t+1|T} - \mu - \boldsymbol{\Phi}\widehat{\boldsymbol{\beta}}_{t|t}\right) \tag{59}$$

$$\mathbf{P}_{t|T} = \mathbf{P}_{t|t} + \mathbf{P}_{t|t}\boldsymbol{\Phi}'\mathbf{P}^{-1}_{t+1|t}\left(\mathbf{P}_{t+1|T} - \mathbf{P}_{t+1|t}\right)\left(\mathbf{P}^{-1}_{t+1|t}\right)'\boldsymbol{\Phi}\left(\mathbf{P}_{t|t}\right)', \tag{60}$$

where $\widehat{\beta}_{T|T}$ and $\mathbf{P}_{T|T}$ are the initial value for the smoothing, obtained from the last iteration of the Kalman filter.

Adrian and Franzoni (2009) use a one-factor version of Equation (44), i.e. they assume that the conditional beta follows a mean-reverting process, with and without conditioning variables such as the term spread. In addition, they also introduce a time-varying unobservable long-run beta $\overline{\boldsymbol{\beta}_t}$ and they consequently add an updating equation for this coefficient. According to Adrian and Franzoni (2009), the measurement update with the Kalman gain provides a realistic representation of the investor's learning process regarding the unknown beta.

From an empirical perspective, Choudhry and Wu (2008), Faff et al. (2000), Mergner (2008), and Nieto et al. (2014) compare different methodologies for estimating time-varying betas. In particular, they compare different multivariate GARCH specifications and Kalman models (Random Walk, Random Coefficients, Mean-Reverting). Overall, the Kalman Random Walk model is considered as the best description of time-varying sectoral betas.

3.6 Autoregressive conditional betas

An alternative to the state space model presented above that also allows a direct specification of dynamic conditional betas has recently been proposed by Darolles et al. (2018). Their model, called CHAR, is a multivariate GARCH model based on the Cholesky decomposition of the $m \times m$ (with $m = N+1$) conditional covariance matrix $\boldsymbol{\Sigma}_t$ of $(\mathbf{x}_t, y_t)'$.

As Pourahmadi (1999), let us consider the Cholesky decomposition of $\boldsymbol{\Sigma}_t$, i.e.

$$\boldsymbol{\Sigma}_t = \boldsymbol{L}_t\boldsymbol{G}_t\boldsymbol{L}_t',$$

where $\boldsymbol{G}_t = \text{diag}(g_{11,t}, \ldots, g_{mm,t})$ and $\boldsymbol{L}_t$ is a lower unitriangular matrix (i.e. triangular with 1's on the diagonal and 0's above the diagonal) with element $\ell_{ij,t}$ at the row i and column j for $i > j$.

Let us now illustrate this decomposition for $m = 3$.

$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21,t} & 1 & 0 \\ l_{31,t} & l_{32,t} & 1 \end{bmatrix} \quad \boldsymbol{G} = \begin{bmatrix} g_{11,t} & 0 & 0 \\ 0 & g_{22,t} & 0 \\ 0 & 0 & g_{33,t} \end{bmatrix}$$

and

$$\boldsymbol{\Sigma} = \begin{bmatrix} g_{11,t} & l_{21,t}g_{11,t} & l_{31,t}g_{11,t} \\ l_{21,t}g_{11,t} & l^2_{21,t}g_{11,t} + g_{22,t} & l_{21,t}l_{31,t}g_{11,t} + l_{32,t}g_{22,t} \\ l_{31,t}g_{11,t} & l_{21,t}h_{31,t}g_{11,t} + l_{32,t}g_{22,t} & l^2_{31,t}g_{11,t} + l^2_{32,t}g_{22,t} + g_{33,t} \end{bmatrix}.$$

Darolles et al. (2018) show that if $\boldsymbol{w}_t \equiv (\mathbf{x}_t, y_t)'$ has mean $\mathbf{0}$,

$$\begin{aligned} w_{i,t} &= \sum_{j=1}^{i-1} \ell_{ij,t}\varepsilon_{j,t} + \varepsilon_{i,t} = \sum_{j=1}^{i-1} \ell_{ij,t} \left(w_{j,t} - \sum_{k=1}^{j-1} \ell_{jk,t} v_{k,t} \right) + \varepsilon_{i,t} \\ &= \sum_{j=1}^{i-1} \beta_{ij,t} w_{j,t} + \varepsilon_{i,t}. \end{aligned}$$

Interestingly, for $i = m$, the m'th equation of the CHAR model is

$$y_t = \sum_{j=1}^{m-1} \beta_{ij,t} x_{j,t} + \varepsilon_{i,t}, \tag{61}$$

which corresponds to Equation (35) when $\alpha_t = 0$, $N = m - 1$ and $\varepsilon_t = \varepsilon_{i,t}$.

Darolles et al. (2018) show that $g_{ii,t}$ is the conditional variance of $w_{i,t}$ and rely on a GARCH model to specify its dynamics. They also propose several specifications for the dynamics of the conditional betas and study the statistical properties of the MLE and Gaussian QML of this model. In their application, they retain the following specification of the conditional betas:

$$\beta_{ij,t} = \beta_{ij} + a_{ij}\varepsilon_{i,t-1}\varepsilon_{j,t-1} + b_{ij}\beta_{ij,t-1}. \tag{62}$$

The main drawback of this model is therefore that it requires estimating the system sequentially because $\beta_{ij,t}$ not only depends on $\varepsilon_{i,t-1}$ but also on $\varepsilon_{j,t-1}$, the error term of the j's (with $j < i$) equation in the Cholesky decomposition. Darolles et al. (2018) also derive stationarity conditions and prove the consistency and the asymptotic normality of the QML estimator of this model.

Building upon the CHAR model, Blasques et al. (2020) propose another model, called the Autoregressive Conditional Beta (ACB) model, which does not require the estimation of the whole system and that outperforms the CHAR specification in the modeling of conditional betas.

Using the same notation as in Equation (35), the ACB model is specified as

$$y_t = \beta_{0,t} x_{0,t} + \sum_{n=1}^{N} \beta_{n,t} x_{n,t} + \varepsilon_t \tag{63}$$

$$\varepsilon_t = \sigma_t z_t, \quad z_t \overset{i.i.d.}{\sim} N(0,1) \tag{64}$$

$$\beta_{i,t} = \beta_i + \sum_{k=1}^{K} \theta_{i,k} Z_{k,t-1} + a_i x_{i,t-1}\varepsilon_{t-1} + b_i \beta_{i,t-1}, \forall i = 0, \ldots, N, \tag{65}$$

where $x_{0,t} = 1\ \forall t$ and σ_t^2 is a GARCH(1,1) model as in (25). Note that $\beta_{0,t}x_{0,t} \equiv \alpha_t$ is a time-varying alpha when $a_0 \neq 0$ and $b_0 \neq 0$ but can be constrained to be a constant (like in the empirical application) by setting $a_0 = b_0 = 0$.

This model is very general because it nests the static Model (31) when $\forall i = 0, \ldots, N$, $K = 0$ and $a_i = b_i = 0$ but also Model (37) with interaction variables when $\theta_{0,k} = 0$ and $\forall k = 0, \ldots, K$, $a_i = b_i = 0$.

Blasques et al. (2020) study the statistical properties of this model (stationarity and invertibility) but also those of the MLE and Gaussian Quasi-Maximum Likelihood estimators and prove convergence and asymptotic normality under mild conditions.

4 Empirical application on REITs

Real Estate Investment Trusts[23] (REITs), which are publicly-traded real estate companies that own and manage commercial or residential real estate, are attractive alternatives to the mainstream investment choice (e.g. stocks and bonds) since they allow investors to easily access real estate investments without directly owning or managing the underlying assets.[24] Moreover, the literature on real estate has shown that the inclusion of REITs within one's portofolio improves the risk-return profile of the portfolio. Compared to other asset classes such as bonds and stocks, they have the characteristics of offering more stable returns and a lower volatility. For the purpose of portfolio diversification, it is important to know how the level of exposure of REITs to both the bond market risk and to the stock market risk varies over time. The aim of this section is thus to perform a comparative analysis of the three most advanced modeling techniques (state space, DCB and ACB) used in estimating the sensitivity of REIT indices to changes in both the bond market and the stock market. Van Nieuwerburgh (2019) argues that a model with a bond market and stock market factor is both the most basic and most natural model of risk for REITs as the bond market beta measures how sensitive REITs are to changes in interest rates and the stock market beta measures how sensitive REITs are to changes in economic activity.[25] A similar model is used by Allen et al. (2000). Moreover, we note that the addition of three Fama-French risk factors (size, value and momentum) to the original two-factor model in the study of Van Nieuwerburgh (2019) leaves the bond and stock market betas almost unchanged. As a consequence, we follow Van Nieuwerburgh (2019) and choose to perform our analysis on the following parsimonious two-factor model:

$$\widetilde{r}_{REIT,t} = \alpha_t + \beta_{B,t}\widetilde{r}_{B,t} + \beta_{M,t}\widetilde{r}_{M,t} + \varepsilon_t, \tag{66}$$

[23]REITs were initially established in 1960 when the U.S. Congress enacted the legislation authorizing their existence. Since then, REITs were authorized in many countries (even though REIT regimes vary country-by-country), including most European countries (e.g. the Netherlands in 1969, Belgium in 1995, France in 2003, Germany, Italy, the United Kingdom in 2007 or Portugal in 2019).

[24]Their revenues are mainly generated from the rents they receive and they do not have to pay any corporate tax in exchange for paying most of their taxable income to shareholders (U.S. REITs must distribute 90% of their taxable income to shareholders through dividend payments for example).

[25]We would probably formulate this a bit differently than Van Nieuwerburgh, arguing that market beta measures how sensitive REITs are to changes in economic activity *in the broad sense*, the variability of stock market returns being more related to the expectation of future profits than the real economic activity per se.

where $\widetilde{r}_{REIT}$ is the excess return on the REIT market, measured by the daily excess return on the FTSE EPRA Nareit index, $\widetilde{r}_B$ is the excess return on the bond market, measured by the daily excess return on the sovereign bond index and $\widetilde{r}_M$ is the excess return on the stock market, measured by the daily excess return on the stock market index. Equation (66) corresponds to the conditional risk factor model that we use to estimate conditional betas on day t from a regression of the daily excess REIT index returns on the excess stock market and bond market returns. In the special case where $\boldsymbol{\beta}_t = (\beta_{B,t}, \beta_{M,t})' = \boldsymbol{\beta}$, the betas are restricted to be constant. Note also that for some models the alpha is allowed to be time-varying as well. However, empirical results (not reported here to save space) suggest that $\alpha_t = \alpha$ $(\forall t)$ once allowing the conditional betas of this two-factor model to be time-varying.

Two strands of literature on REIT conditional betas are particularly relevant to our study. The first one considers the exposure of REITs to both interest-rate risk and stock-market risk. Flannery and James (1984) put the emphasis on the fact that firms holding financial assets should be more sensitive to interest-rate risk. Allen et al. (2000) put forward four reasons why equity and mortgage REITs may be affected by changes in interest rates. First, REITs rely heavily on debt so that an increase in interest rates may dampen demand and have a negative impact on valuations (and vice versa). Second, an increase in interest rates may also translate into a higher cost of debt financing. Third, such an increase may result in a higher required rate of return by investors. And fourth, it may raise the cost of present development and refurbishment projects.

A second strand of literature considers potential regime shifts in market betas. Willard and Youguo (1991) find a decline in equity REIT betas over the period 1974-1983. Liang et al. (1995) in a similar study find that the market beta of equity REITs is rather stable over time while the market beta of mortgage REITS declined substantially over the period 1973-1989. However, Chiang et al. (2005) find that when using the Fama-French three-factor model, the declining trend in equity REIT betas evaporates. Finally, Glascock (1991) tests for changes in the market beta of a REIT portfolio during bull and bear markets and finds that the beta behaves procyclically.

Despite the numerous empirical applications focusing on REITs, only a few papers examine how to best model the market beta of REITs. We can mention the papers by Zhou (2013) and Altınsoy et al. (2010). Zhou (2013) compares five modeling techniques in the estimation of the conditional beta of REITs: rolling regression, dynamic conditional correlation (DCC) GARCH model, Schwert and Seguin model, state space model and Markov-switching model. In the same way, the study of Altınsoy et al. (2010) is based on the estimation of the conditional beta of Turkish REITs with a comparison of modeling techniques. Compared to the existing literature, the contribution of our study is fourfold : first we attempt to analyze how to model the conditional beta of REITs focusing on the two largest REIT markets (i.e. the U.S. and European REIT markets), which allows us to compare both markets. Second, we extend the spectrum of modeling techniques by focusing on the most advanced conditional beta modeling techniques, that is to say the state space model, DCB and ACB modeling techniques. Third, we investigate the time variability of betas in a two-factor model and we introduce within this model exogenous variables that may affect the evolution of the bond market beta and the stock market beta. Finally, in addition to an in-sample beta estimation, we extend our analysis to an out-of-sample beta forecasting exercise.

4.1 Data and sample

We focus on both the U.S. and developed Europe[26] REIT markets, proxied by the daily excess log-returns on the FTSE EPRA Nareit United States USD Total Return Index and the FTSE EPRA Nareit Developed Europe EUR Total Return Index. The FTSE EPRA Nareit data are from Thomson Reuters Eikon.

We explore the sensitivity of the two REIT indices to (1) the bond market factor as proxied by the daily excess log-returns on the S&P U.S. Treasury Bond Index and the S&P Eurozone Developed Sovereign Bond Index respectively, and to (2) the stock market factor proxied by the daily excess log-returns on the S&P500 index and the EUROSTOXX600 index respectively. The bond market data are obtained from Standard and Poor's and the stock market data are from Thomson Reuters Eikon.

The risk-free rate is the one-month T-bill rate for the U.S. and the one-month Euribor rate for developed Europe computed on a daily basis. The data are from the Federal Reserve Economic Data (FRED) and the European Money Markets Institute (EMMI) respectively.

Assuming that beta can be influenced by risk aversion, we use three risk aversion indicators commonly used in the literature (Bank, 2007) as exogenous variables that may affect the evolution of the bond market beta and the stock market beta. In particular, we use implied volatility as an indicator of the volatility that is expected by the market, the Ted Spread as an indicator of credit risk in the interbank money market, and the High Yield Option-Adjusted Spread as an indicator of credit risk.

The volatility indices, i.e. the VIX for the U.S and the VSTOXX for developed Europe, respectively measure the 30-day expected volatility of the U.S. stock market based on the S&P500 option prices and the 30-day expected volatility of the European stock market based on the EUROSTOXX50 option prices. We hereafter both call them VIX for simplicity. The Ted Spread is the difference between the three-month Treasury bill and the three-month USD LIBOR rate.[27] The US High Yield Index Option-Adjusted Spread is the difference between a computed option-adjusted spread (OAS) index of all bonds in the Bank of BofAML US High Yield Master II Index and a spot Treasury curve. As for the EUR High Yield Index Option-Adjusted Spread of BofAML, it is the European equivalent of the U.S. High Yield Index Option-Adjusted Spread.

The full sample runs from October 02, 2009 to October 01, 2019, which amounts to 2,554 daily returns.

[26]Based on the FTSE EPRA Nareit Developed Europe EUR Total Return Index, developed Europe includes the following countries: Austria, Belgium, Finland, France, Germany, Ireland, Italy, the Netherlands, Norway, Spain, Sweden, Switzerland and the United Kingdom.

[27]While the Ted Spread originally measures interbank risk in the U.S., we believe that this indicator may also be relevant for Europe as well, the U.S. being the leading engine of credit worldwide.

4.2 Empirical results

The aim of our empirical study is to compare the performance of the competing models from two perspectives: in-sample beta estimates on the one hand and out-of-sample beta forecasts on the other hand. The in-sample analysis is meant to assess how well the different models fit the data while the out-of-sample analysis is useful in assessing what modeling technique provides the best beta forecasts in a tracking exercise. A better beta forecast can then be used as an input within many financial applications.

We use the first 2,304 observations of our sample as the in-sample period and the remaining 250 observations as the out-of-sample period. Having performed a sensitivity analysis on the same sample by increasing the number of out-of-sample period observations (to 500 and 750 observations respectively) and having found that the results were qualitatively the same, we do not report them to save space.

4.2.1 In-sample estimates

Seven competing models have been estimated using the Ox programming language (Doornik, 2012) and the G@RCH 8.0 software (Laurent, 2018) to obtain the conditional bond market betas ($\beta_{B,t}$) and the conditional stock market betas ($\beta_{M,t}$) for both the U.S and developed Europe. We estimate the following two-factor model:

$$\widetilde{r}_{REIT,t} = \alpha_t + \beta_{B,t}\widetilde{r}_{B,t} + \beta_{M,t}\widetilde{r}_{M,t} + \varepsilon_t. \tag{67}$$

The seven competing models are the following:

1. OLS model:

$$\begin{aligned} &\alpha_t = \alpha,\ \beta_{B,t} = \beta_B,\ \beta_{M,t} = \beta_M\ \forall t, \\ &\varepsilon_t \overset{i.i.d.}{\sim} N(0, \sigma^2). \end{aligned}$$

2. Univariate GARCH model:

$$\begin{aligned} &\alpha_t = \alpha,\ \beta_{B,t} = \beta_B,\ \beta_{M,t} = \beta_M\ \forall t, \\ &\varepsilon_t = \sigma_t z_t, \quad z_t \overset{i.i.d.}{\sim} N(0, 1), \\ &\sigma_t^2 = \lambda_0 + \lambda_1 \sigma_{t-1}^2 + \lambda_2 \varepsilon_{t-1}^2. \end{aligned}$$

3. Univariate GARCH model with interaction variables (GARCH-Z):

$$\begin{aligned} \alpha_t &= \alpha\,\forall t, \\ \beta_{B,t} &= c_B + \theta_{B,TED} TED_{t-1} + \theta_{B,HY} HY_{t-1}, \\ \beta_{M,t} &= c_M + \theta_{M,VIX} VIX_{t-1}, \\ \varepsilon_t &= \sigma_t z_t, \quad z_t \overset{i.i.d.}{\sim} N(0, 1), \\ \sigma_t^2 &= \lambda_0 + \lambda_1 \sigma_{t-1}^2 + \lambda_2 \varepsilon_{t-1}^2, \end{aligned}$$

where TED_t, HY_t, and VIX_t are respectively the TED spread, the High Yield Index Option-Adjusted Spread, and the VIX index, $\theta_{B,TED}$ and $\theta_{B,HY}$ are the coefficients of the two interaction variables entering into the conditional betas of the bond market, and $\theta_{M,VIX}$ is the coefficient of the interaction variable entering into the conditional betas of the stock market.

4. State Space Model (SSM):

$$\begin{aligned} \alpha_t &= \alpha \, \forall t, \\ \beta_{B,t} &= \beta_{B,t-1} + u_{B,t}, \\ \beta_{M,t} &= \beta_{M,t-1} + u_{M,t}, \end{aligned}$$

$$\begin{pmatrix} u_{B,t} \\ u_{M,t} \\ \varepsilon_t \end{pmatrix} | \mathbf{Y}_{t-1}, \mathbf{X}_t \overset{i.i.d}{\sim} N\left(\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_B^2 & 0 & 0 \\ 0 & \sigma_M^2 & 0 \\ 0 & 0 & \sigma_\varepsilon^2 \end{pmatrix} \right) \right).$$

5. Dynamic Conditional Beta Model (DCB):

$$\begin{pmatrix} \mathbf{x}_t \\ y_t \end{pmatrix} | \mathcal{F}_{t-1} \sim N\left(\begin{pmatrix} \mathbf{0} \\ 0 \end{pmatrix}, \mathbf{\Sigma}_t \equiv \begin{pmatrix} \mathbf{\Sigma}_{\mathbf{xx},t} & \mathbf{\Sigma}_{\mathbf{x}y,t} \\ \mathbf{\Sigma}_{y\mathbf{x},t} & \mathbf{\Sigma}_{yy,t} \end{pmatrix} \right),$$

$$\alpha_t = \alpha \forall t,$$

$$\widehat{\boldsymbol{\beta}}_t^{DCB} \equiv (\hat{\beta}_{B,t}, \hat{\beta}_{M,t})' = \mathbf{\Sigma}_{\mathbf{xx},t}^{-1} \mathbf{\Sigma}_{\mathbf{x}y,t}.$$

where $\mathbf{x}_t = (\widetilde{r}_{B,t}, \widetilde{r}_{M,t})', y_t = \widetilde{r}_{REIT,t}$ and $\mathbf{\Sigma}_t$ is specified as a DCC-GARCH(1,1) model.

6. Autoregressive Conditional Beta (ACB):

$$\begin{aligned} \alpha_t &= \alpha \, \forall t, \\ \beta_{B,t} &= c_B + a_B x_{i,t-1} \varepsilon_{t-1} + b_B \beta_{B,t-1}, \\ \beta_{M,t} &= c_M + a_M x_{i,t-1} \varepsilon_{t-1} + b_M \beta_{M,t-1}, \\ \varepsilon_t &= \sigma_t z_t, \quad z_t \overset{i.i.d.}{\sim} N(0,1). \end{aligned}$$

7. Autoregressive Conditional Beta with interaction variables (ACB-Z):

$$\begin{aligned} \alpha_t &= \alpha \, \forall t, \\ \beta_{B,t} &= c_B + a_B x_{i,t-1} \varepsilon_{t-1} + b_B \beta_{B,t-1}, \\ \beta_{M,t} &= c_M + a_M x_{i,t-1} \varepsilon_{t-1} + b_M \beta_{M,t-1} + \theta_{M,VIX} VIX_{t-1}, \\ \varepsilon_t &= \sigma_t z_t, \quad z_t \overset{i.i.d.}{\sim} N(0,1), \end{aligned} \tag{68}$$

where $\theta_{B,VIX}$ is the coefficient of the interaction variable entering into the conditional betas of the stock market.[28]

[28] The other two coefficients entering into the conditional betas of the bond market (i.e. $\theta_{B,TED}$ and $\theta_{B,HY}$) are found to be insignificant so that the corresponding variables are removed from the model.

To ease the presentation of the results throughout the rest of the paper, we use GARCH, GARCH-Z, DCB, SSM, ACB and ACB-Z to respectively denote the GARCH model, the GARCH model with interaction variables, the dynamic conditional beta model, the state space model, the autoregressive conditional beta model and the autoregressive conditional beta model with interaction variables.

The GARCH-Z model corresponds to a GARCH model with interaction variables in the conditional mean or equivalently an ACB model with no dynamics (i.e. $a_i = b_i = 0$, $\forall i$) but with explanatory variables. SSM is a state space model without dynamics in the intercept but in which the two slope coefficients follow a random walk. The ACB and ACB-Z models are two ACB models, but exogenous explanatory variables are only included in the latter model. Note also that a GARCH specification is included in the residuals of all the models except for the OLS and SSM.

Estimation results of the different models are reported in Tables 1 and 2, respectively for the U.S. and developed Europe.

We do report the estimated parameters for the sake of completeness but it is hard to draw any conclusion from them. To compare the models in-sample, we rely on both the log-likelihood and the Bayesian Information Criterion (BIC). Table 3 reports the number of parameters, the log-likelihood as well as the BIC of all the models at the exception of the DCB.[29] Results suggest that the GARCH-Z, ACB and ACB-Z models always outperform the other models for both the U.S. and developed Europe. Interestingly, when comparing the GARCH-Z model to the GARCH model based on both the BIC and a likelihood ratio test, we find that results are clearly in favor of the GARCH-Z model, suggesting that the risk factors used to capture the dynamics in the conditional betas are relevant. Results reported in Tables 1 and 2 indeed suggest that these three variables help predict the dynamics of the two conditional betas in the GARCH-Z model. However, although the ACB model does not rely on these exogenous factors, it further improves the estimation of the dynamics of the two conditional betas.

According to the BIC, the best model is the ACB-Z model, i.e. an ACB model with the additional VIX explanatory variable in the conditional beta of the stock market. Note however that the SSM model imposes the variance of the error term to be homoscedastic, which certainly explains why it performs so badly according to the BIC. Extending the SSM model by accounting for GARCH effects in the residuals is therefore desirable but beyond the scope of this paper.

The estimated conditional betas $\beta_{B,t}$ and $\beta_{M,t}$ are plotted in Figures 1 and 2, respectively for the U.S and developed Europe. Each graph contains the estimated betas for the seven competing models of our study. Both graphs reveal large fluctuations over time regarding the exposure of REITs to the two risk factors of our model.

The reading of these graphs leads to various observations. First, we observe that both the bond market beta and the stock market beta are not constant over time but time vary-

[29] Recall that the DCB model requires estimating an MGARCH model (i.e. a DCC-GARCH model in our case) so that the obtained log-likelihood is for the joint distribution of the three series $(\widetilde{r}_{REIT,t}, \widetilde{r}_{B,t}, \widetilde{r}_{M,t})'$ and not for $\widetilde{r}_{REIT,t}$ given $\widetilde{r}_{B,t}$ and $\widetilde{r}_{M,t}$ as for the other six models.

Table 1: Parameter estimates for the United States

	α	β_B	β_M									
OLS	-0.018	1.243	1.081									
	(0.017)	(0.126)	(0.031)									
	λ_0	λ_1	λ_2	α	β_B	β_M						
GARCH	0.011	0.064	0.919	-0.014	1.463	0.970						
	(0.006)	(0.016)	(0.0232)	(0.015)	(0.121)	(0.030)						
	λ_0	λ_1	λ_2	c_B	c_M	a_B	a_M	b_B	b_M	$\theta_{B,TED}$	$\theta_{B,HY}$	$\theta_{M,VIX}$
GARCH-Z	0.007	0.041	0.947	2.887	0.524					-4.393	27.976	2.104
	(0.004)	(0.012)	(0.017)	(0.496)	(0.073)					(0.878)	(8.768)	(0.363)
ACB	0.007	0.034	0.950	0.004	0.002	-0.164	0.016	0.999	0.997			
	(0.005)	(0.012)	(0.020)	(0.000)	(0.000)	(0.007)	(0.001)	(0.000)	(0.000)			
ACB-Z	0.007	0.037	0.947	0.005	0.002	-0.156	0.015	0.997	0.990			0.032
	(0.005)	(0.012)	(0.021)	(0.000)	(0.000)	(0.004)	(0.001)	(0.000)	(0.000)			(0.000)
	σ_e	σ_B	σ_M	σ_η								
SSM	0.713	0.061	0.017	0.000								

The table reports the parameter estimates of the different models. Standard errors are in parentheses. $GARCH - Z$ corresponds to the GARCH model with interaction variables. $ACB - Z$ corresponds to the ACB model with both dynamics and an explanatory variable in the market beta. SSM corresponds to the state space model. Parameters of the DCC model used to get the conditional betas of the DCB are not reported.

Table 2: Parameter estimates for developed Europe

	α	β_B	β_M									
OLS	0.008	0.478	0.808									
	(0.012)	(0.083)	(0.019)									
	λ_0	λ_1	λ_2	α	β_B	β_M						
GARCH	0.010	0.052	0.921	0.016	0.448	0.783						
	(0.007)	(0.022)	(0.041)	(0.012)	(0.075)	(0.021)						
	λ_0	λ_1	λ_2	c_B	c_M	a_B	a_M	b_B	b_M	$\theta_{B,TED}$	$\theta_{B,HY}$	$\theta_{M,VIX}$
GARCH-Z	0.007	0.043	0.936	0.515	0.487					-1.158	18.489	1.144
	(0.004)	(0.017)	(0.028)	(0.214)	(0.052)					(0.357)	(6.360)	(0.224)
ACB	0.009	0.058	0.913	0.007	0.021	0.201	0.023	0.983	0.971			
	(0.009)	(0.035)	(0.058)	(0.006)	(0.011)	(0.153)	(0.01)	(0.014)	(0.014)			
ACB-Z	0.008	0.046	0.930	0.009	0.058	0.263	0.015	0.979	0.863			0.192
	(0.005)	(0.020)	(0.032)	(0.005)	(0.031)	(0.151)	(0.007)	(0.011)	(0.070)			(0.107)
	σ_e	σ_B	σ_M	σ_η								
SSM	0.551	0.093	0.031	0.0000								

See the note of Table 1.

Table 3: Comparison of the different models based on log-likelihood

		United States		Developed Europe	
	#para	Log-lik	BIC	Log-lik	BIC
OLS	4	-2,726.81	2.3805	-2,046.09	1.7896
GARCH	6	-2,588.30	2.2670	-1,972.63	1.7325
GARCH-Z	9	-2,502.28	2.2033	-1,930.44	1.7067
SSM	4	-2,531.73	2.2010	-1,996.49	1.7364
ACB	10	-2,444.96	2.1560	-1,939.16	1.7169
ACB-Z	11	-2,442.61	2.1582	-1,925.20	1.7089

The table reports the number of parameters (#para), log-likelihood (Log-lik) and Bayesian Information Criterion (BIC) of all the models (at the exception of the DCB) for both the United States and developed Europe.

ing (contrary to the assumption made when using the OLS approach), which confirms the appropriateness of conditional CAPM modeling. Second, the stock market beta for the U.S. and to a lesser extent for developed Europe is on a declining track over the considered period, implying that the sensitivity of the REIT sector to the overall equity market is decreasing and can be interpreted as a sign of a maturing REIT market, which is consistent with the out-of-sample results of Zhou (2013), even though his study ends in 2011. The picture is however different regarding the bond market beta of both the U.S. and developed Europe since the bond market beta of both areas is rather on a rising track over the same period. This difference justifies both using a two-factor model and comparing the U.S. to Europe. Third, while we observe that all the dynamic methods (DCB, ACB, SSM) present similar dynamics, we also note that the conditional betas of the DCB model are far more erratic than those of the SSM and ACB models. Indeed, the conditional betas obtained with the DCB model are very choppy over the estimation period while those obtained with the SSM and ACB models are much smoother on average. Fourth, we observe periods where the betas obtained with the different models are very close to one another and periods where the betas are very far from one another so that one can expect differences between the different models in terms of performance. Finally, the conditional bond and stock market betas filtered with the SSM model are smoother than those obtained with the other models of our comparative study. However, this effect can be explained by the fact a random walk is imposed in this model. Finally, we find that the correlation between the betas of the SSM model and the betas of the ACB model is very high (0.77 for the developed Europe bond market beta, 0.71 for the developed Europe stock market beta, 0.89 for the U.S. bond market beta, 0.94 for the U.S. stock market beta) and we note that the correlation remains high when we add explanatory variables to the betas of the ACB model.

4.2.2 Out-of-sample estimates

The aim of this section is to illustrate the usefulness of the competing models in a portfolio and risk management exercice. Since realized betas are not observed, it is impossible to judge the quality of the models by looking at the forecasting errors of the conditional betas.

Instead, following Engle (2016) and Darolles et al. (2018), we perform a tracking exercise that consists in taking a position at time t in the two considered factors (bond and market) whose weights are the one-step ahead forecasts of the corresponding conditional betas.

For each model, the conditional betas forecasts are therefore used to construct a hedging portfolio. The returns on this portfolio are obtained using the conditional betas forecasts, i.e.

$$Z_{REIT,t+1|t} = \beta_{B,t+1|t}\widetilde{r}_{B,t+1} + \beta_{M,t+1|t}\widetilde{r}_{M,t+1}, \tag{69}$$

where $\widetilde{r}_{B,t+1}$ and $\widetilde{r}_{M,t+1}$ are the realized excess log-returns of the two factors at time $t+1$ while $\beta_{B,t+1|t}$ and $\beta_{M,t+1|t}$ are the one-step-ahead forecasts of the conditional betas obtained at the end of day t.

This hedging portfolio can be interpreted as a portfolio invested in the risk factors and which optimally tracks the corresponding REIT returns. It is a hedging portfolio in the sense that it can be sold short to hedge the main risks of a given portfolio. In this asset pricing context, expected returns on any asset are linear in the betas and only depend upon the risk premiums embedded in the factors. In other words, there is no alpha or intercept in (69).

For both the U.S. and developed Europe, we compute the ex-post tracking errors as follows:

$$TE_{t+1|t} = \widetilde{r}_{REIT,t+1} - Z_{REIT,t+1|t} \tag{70}$$

and we look for the model that has the smallest sample mean square error (MSE) and mean absolute deviation (MAD) over the 250 values of the tracking errors using the Model Confidence Set approach of Hansen et al. (2011). Models are reevaluated every 25 steps so that estimated parameters are kept constant to produce 25 one-step-ahead forecasts of the conditional betas before being updated.

The forecasted one-steap-ahead conditional betas $\beta_{B,t+1|t}$ and $\beta_{M,t+1|t}$ are plotted in Figures 3 and 4, respectively for the U.S and developed Europe. Each graph contains the forecasted betas for the seven competing models. Both graphs again reveal large fluctuations over time regarding the exposure of REITs to the two risk factors of our model and we observe large discrepancies between the competing methods.

Table 4 reports the MSE and the MAD as well as the results of the MCS test with an MSE loss function, a significance level of 5%, and 10,000 bootstrap samples (with a block length of 5 observations). Models highlighted with the symbol ✓ are contained in the model confidence set (or set of superior models) when relying either on the MSE or MAD loss function (i.e. results are identical for both loss functions). The ACB, ACB-Z and SSM models clearly outperform the other models. The ACB model has the lowest MSE for both series although its MSE is not statistically different from those of the ACB-Z and SSM at the 5% nominal size according to the MCS test. Interestingly, the models with constant betas clearly underperform the models with time-varying betas. Furthermore, the conditional

Table 4: Comparison of out-of-sample results

Panel A: United States	MSE	MAD	MCS
OLS	0.8463	0.6670	
GARCH	0.7381	0.6240	
GARCH-Z	0.7273	0.6175	
DCB	0.6808	0.6040	
SSM	0.6336	0.5694	✓
ACB	0.6071	0.5769	✓
ACB-Z	0.6018	0.5723	✓
Panel B: Developed Europe	**MSE**	**MAD**	**MCS**
OLS	0.4758	0.5391	
GARCH	0.4663	0.5344	
GARCH-Z	0.4426	0.5219	
DCB	0.4504	0.5311	
SSM	0.4279	0.5201	✓
ACB	0.4063	0.5042	✓
ACB-Z	0.4074	0.5058	✓

The table reports three evaluation criteria: MSE (mean square error), MAD (mean absolute deviation) and MCS (model confidence set). Models highlighted with the symbol ✓ are contained in the model confidence set (or set of superior models).

betas of the DCB model are found to be much more volatile than those of the ACB models and the SSM (which would imply higher transaction costs). Finally, it appears that the DCB model significantly underperforms in this tracking exercise.

5 Conclusion

In this chapter, we review the different time series models used to estimate static and time-varying betas and then compare the performance of the standard static beta models (i.e. OLS and GARCH models) to the most advanced conditional beta modeling techniques that are the state space model, the dynamic conditional beta model and the autoregressive conditional beta model (with or without additional exogenous variables). The analysis is performed on the two largest REIT markets in the world, that is to say the U.S. and developed Europe

REIT markets, over the period 2009-2019. In particular, we investigate the time variability of betas in a two-factor model, where $\beta_{B,t}$ and $\beta_{M,t}$ are respective measures of the sensitivity of the REIT index to changes in the bond market and the stock market. Assuming that beta may depend on risk aversion, we use three risk aversion indicators as exogenous variables that may affect the evolution of the bond market beta and the stock market beta.

Based on the employed evaluation criteria, we evaluate the performance of the seven competing models both in terms of in-sample estimates and through an out-of-sample tracking exercise. Results reveal several meaningful findings. First, dynamic models clearly outperform static models both in- and out-of-sample, meaning that both the bond market beta and the stock market beta are not constant over time but time varying, which gives convincing arguments for modeling conditional, instead of static, betas. Second, the autoregressive conditional beta model with additional exogenous variables outperforms the other techniques for both the U.S. and developed Europe, followed by the autoregressive conditional beta model without additional variables and the state space model. The dynamic conditional beta model delivers an unsatisfactory out-of-sample predictive performance. Finally, the inclusion of risk aversion indicators as exogenous variables into the ACB model (but also into the GARCH model) helps improve the prediction of betas.

These results can be used in many financial situations, like for example for estimating the cost of capital with the aim of capital budgeting involving REITs, for evaluating the performance of REIT portfolios or for decisions concerning asset allocation and portfolio diversification.

References

Adrian, T., Franzoni, F., 2009. Learning about beta: Time-varying factor loadings, expected returns, and the conditional CAPM. Journal of Empirical Finance 16, 537–556.

Alexander, G. J., Chervany, N. L., 1980. On the estimation and stability of beta. Journal of Financial and Quantitative Analysis 15 (1), 123–137.

Allen, M. T., Madura, J., Springer, T. M., 2000. Reit characteristics and the sensitivity of reit returns. The Journal of Real Estate Finance and Economics 21 (2), 141–152.

Altınsoy, G., Erol, I., Yıldırak, S., 2010. Time-varying beta risk of turkish real estate investment trusts. Middle East Technical University Studies in Development 37 (2), 83–114.

Andersen, T. G., Bollerslev, T., Diebold, F. X., Labys, P., 2003. Modelling and forecasting realized volatility. Econometrica 71 (2), 579–625.

Andersen, T. G., Bollerslev, T., Diebold, F. X., Wu, G., 2006. Realized beta: Persistence and predictability. In: Fomby, T., Terrell, D. (Eds.), Econometric Analysis of Financial and Economic Time Series. Vol. 20 of Advances in Econometrics. Emerald Group Publishing Limited, Bingley, pp. 1–39.

Bali, Turan G.; Engle, R. F., 2010. Resurrecting the conditional CAPM with dynamic conditional correlations. SSRN Electronic Journal.

Bank, E. C., 2007. Measuring investors' risk appetite. Financial Stability Review, 166–171.

Barndorff-Nielsen, O. E., Shephard, N., 2004. Econometric analysis of realized covariation: High frequency based covariance, regression, and correlation in financial economics. Econometrica 72 (3), 885–925.

Bauwens, L., Laurent, S., Rombouts, J., 2006. Multivariate GARCH models: a survey. Journal of Applied Econometrics 21, 79–109.

Blasques, F., Francq, C., Laurent, S., 2020. Dynamic conditional betas, working paper.

Bollerslev, T., Engle, R. F., Wooldridge, J. M., 1988. A capital asset pricing model with time-varying covariances. Journal of Political Economy 96 (1), 116–131.

Bollerslev, T., et al., 1990. Modelling the coherence in short-run nominal exchange rates: a multivariate generalized arch model. Review of Economics and statistics 72 (3), 498–505.

Boudt, K., Lunde, A., Laurent, S., Quaedvlieg, R., Sauri, O., 2017. Positive semidefinite integrated covariance estimation,factorizations and asynchronicity. Journal of Econometrics 196 (2), 347–367.

Brooks, C., 2014. Introductory econometrics for finance. Cambridge university press.

Brooks, R., Faff, R., Lee, J., 1992. The form of time variation of systematic risk: Testing for beta instability. Pacific-Basin Finance Journal 2, 191–198.

Brooks, R. D., Faff, R. W., Lee, J. H., 1994. Beta stability and portfolio formation. Pacific-Basin Finance Journal 2 (4), 463–479.

Carhart, M. M., 1997. On persistence in mutual fund performance. The Journal of Finance 52 (1), 57–82.

Chiang, K. C., Lee, M.-L., Wisen, C. H., 2005. On the time-series properties of real estate investment trust betas. Real Estate Economics 33 (2), 381–396.

Choudhry, T., Wu, H., 2008. Forecasting ability of GARCH vs Kalman filter method: evidence from daily uk time-varying beta. Journal of Forecasting 27, 670–689.

Cisse, M., Konte, M., Toure, M., Assani, S., 2019. Contribution to the valuation of brvm's assets: A conditional CAPM approach. Journal of Risk and Financial Management 12, 27.

Cochrane, J., 2005. Asset pricing. Princeton Univ. Press, Princeton.

Cochrane, J. H., Culp, C. L., 2003. Equilibrium asset pricing and discount factors: Overview and implications for derivatives valuation and risk management. In: Hughston, L. P. (Ed.), Modern Risk Management: A History, introduced by Peter Field. Risk Books London, UK, Ch. 5, pp. 57–92.

Darolles, S., Francq, C., Laurent, S., 2018. Asymptotics of cholesky GARCH models and time-varying conditional betas. Journal of Econometrics 204 (2), 223–247.

DeJong, D. V., Collins, D. W., 1985. Explanations for the instability of equity beta: Risk-free rate changes and leverage effects. Journal of Financial and Quantitative Analysis 20 (1), 73–94.

Doornik, J., 2012. Object-oriented matrix programming using Ox. Timberlake Consultants Press.

Durbin, J., Koopman, S. J., 2001. Time Series Analysis by State Space Methods (Oxford Statistical Science Series). Oxford University Press.

Engle, R. F., 2016. Dynamic conditional beta. Journal of Financial Econometrics 14 (4), 643–667.

Fabozzi, F. J., Francis, J. C., 1977. Stability tests for alphas and betas over bull and bear market conditions. The Journal of Finance 32 (4), 1093–1099.

Faff, R. W., Hillier, D., Hillier, J., 2000. Time varying beta risk: An analysis of alternative modelling techniques. Journal of Business Finance and Accounting 27, 523–554.

Fama, E. F., French, K. R., 1993. Common risk factors in the returns on stocks and bonds. Journal of Financial Economics 33 (1), 3–56.

Fama, E. F., French, K. R., 2004. The capital asset pricing model: Theory and evidence. Journal of Economic Perspectives 18 (3), 25–46.

Fama, E. F., French, K. R., 2015. A five-factor asset pricing model. Journal of Financial Economics 116 (1), 1–22.

Fama, E. F., MacBeth, J. D., 1973. Risk, return, and equilibrium: Empirical tests. Journal of political economy 81 (3), 607–636.

Ferson, W. E., 2003. Tests of multifactor pricing models, volatility bounds and portfolio performance. In: Constantinides, G., Harris, M., Stulz, R. (Eds.), Handbook of the Economics of Finance. Financial Markets and Asset Pricing. Vol. 1. Elsevier, pp. 743–802.

Ferson, W. E., Jagannathan, R., 1996. Econometric evaluation of asset pricing models. In: Maddala, G., Rao, C. (Eds.), Handbook of Statistics. Statistical Methods in Finance. Vol. 14. Elsevier, pp. 1–33.

Fisher, L., Kamin, J. H., 1985. Forecasting systematic risk: Estimates of “raw” beta that take account of the tendency of beta to change and the heteroskedasticity of residual returns. Journal of Financial and Quantitative Analysis 20 (2), 127–149.

Flannery, M. J., James, C. M., 1984. The effect of interest rate changes on the common stock returns of financial institutions. The Journal of Finance 39 (4), 1141–1153.

Gagliardini, P., Ossola, E., Scaillet, O., 2016. Time-varying risk premium in large cross-sectional equity data sets. Econometrica 84 (3), 985–1046.

Glascock, J. L., 1991. Market conditions, risk, and real estate portfolio returns: some empirical evidence. The Journal of Real Estate Finance and Economics 4 (4), 367–373.

Hamilton, J. D., 1994. Time Series Analysis. Princeton University Press.

Hansen, L. P., 1982. Large sample properties of generalized method of moments estimators. Econometrica 50 (4), 1029–1054.

Hansen, P., Lunde, A., Nason, J., 2011. The model confidence set. Econometrica 79, 453–497.

Hansen, P. R., Lunde, A., Voev, V., 2014. Realized beta GARCH: A multivariate GARCH model with realized measures of volatility. Journal of Applied Econometrics 29 (5), 774–799.

Harvey, C. R., Liu, Y., Zhu, H., 10 2015. ...and the cross-section of expected returns. The Review of Financial Studies 29 (1), 5–68.

Harvey, C. R., Siddique, A., 2000. Conditional skewness in asset pricing tests. The Journal of Finance 55, 1263–1295.

Huang, Peng; Hueng, C. J., 2009. Interest-rate risk factor and stock returns: a time-varying factor-loadings model. Applied Financial Economics 19, 1813–1824.

Laurent, S., 2018. G@RCH 8. Estimating and Forecasting GARCH Models. Timberlake Consultants Ltd.

Lewellen, J., Nagel, S., 2006. The conditional CAPM does not explain asset-pricing anomalies. Journal of Financial Economics 82, 289–314.

Liang, Y., Prudential, W., Webb, J., 1995. Intertemporal changes in the riskiness of reits. Journal of Real Estate Research 10 (4), 427–443.

Linnainmaa, J. T., Roberts, M. R., 04 2018. The history of the cross-section of stock returns. The Review of Financial Studies 31 (7), 2606–2649.

Lintner, J., 1965. The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets. Review of Economics and Statistics 47, 13–37.

Mergner, Sascha; Bulla, J., 12 2008. Time-varying beta risk of pan-european industry portfolios: A comparison of alternative modeling techniques. European Journal of Finance 14, 771–802.

Merton, R. C., 1973. An intertemporal capital asset pricing model. Econometrica 41 (5), 867–887.

Moryson, M., 1998. Testing for Random Walk Coefficients in Regression and State Space Models. Contributions to Statistics. Physica-Verlag Heidelberg.

Nieto, B., Susan, O., Ainoha, Z., 1 2014. Time-varying market beta: Does the estimation methodology matter? SORT-Statistics and Operations Research Transactions 38 (1), 13–42.

Ohlson, J., Rosenberg, B., 1982. Systematic risk of the crsp equal-weighted common stock index: A history estimated by stochastic-parameter regression. Journal of Business, 121–145.

Patton, A. J., Verardo, M., 2012. Does beta move with news? firm-specific information flows and learning about profitability. The Review of Financial Studies 25 (9), 2789–2839.

Phelan, G., Toda, A. A., 2015. On the robustness of theoretical asset pricing models. Tech. rep., Department of Economics, Williams College.

Pourahmadi, M., 1999. Joint mean-covariance models with applications to longitudinal data: Unconstrained parameterisation. Biometrika 86 (3), 677–690.

Reyes, M. G., 1999. Size, time-varying beta, and conditional heteroscedasticity in uk stock returns. Review of Financial Economics 8 (1), 1–10.

Roll, R., 1977. A critique of the asset pricing theory's tests part i: On past and potential testability of the theory. Journal of Financial Economics 4, 0–176.

Ross, S. A., 1976. The arbitrage theory of capital asset pricing. Journal of Economic Theory 13, 341–360.

Schadt, W. E. F. R. W., 1996. Measuring fund strategy and performance in changing economic conditions. The Journal of Finance 51, 425–461.

Schwert, G. W., Seguin, P. J., 1990. Heteroskedasticity in stock returns. The Journal of Finance 45 (4), 1129–1155.

Shanken, J., 1990. Intertemporal asset pricing: An empirical investigation. Journal of Econometrics 45, 99–120.

Sharpe, W. F., 1964. Capital asset prices: A theory of market equilibrium under conditions of risk. The Journal of Finance 19 (3), 425–442.

Smith, P., Wickens, M., 2002. Asset pricing with observable stochastic discount factors. Journal of Economic Surveys 16, 397–446.

Sunder, S., 1980. Stationarity of market risk: Random coefficients tests for individual stocks. The Journal of Finance 35 (4), 883–896.

Todorov, V., Bollerslev, T., 2010. Jumps and betas: A new framework for disentangling and estimating systematic risks. Journal of Econometrics 157 (2), 220–235.

Van Nieuwerburgh, S., 2019. Why are reits currently so expensive? Real Estate Economics 47 (1), 18–65.

Wang, R. J. Z., 1996. The conditional CAPM and the cross-section of expected returns. The Journal of Finance 51, 3–53.

Willard, M., Youguo, L., 1991. An examination of the small-firm effect within the reit industry. Journal of Real Estate Research 6 (1), 9–17.

Zhou, J., 2013. Conditional market beta for reits: a comparison of modelling techniques. Economic Modelling 30, 196–204.

Figure 1: In-sample conditional betas for the U.S.

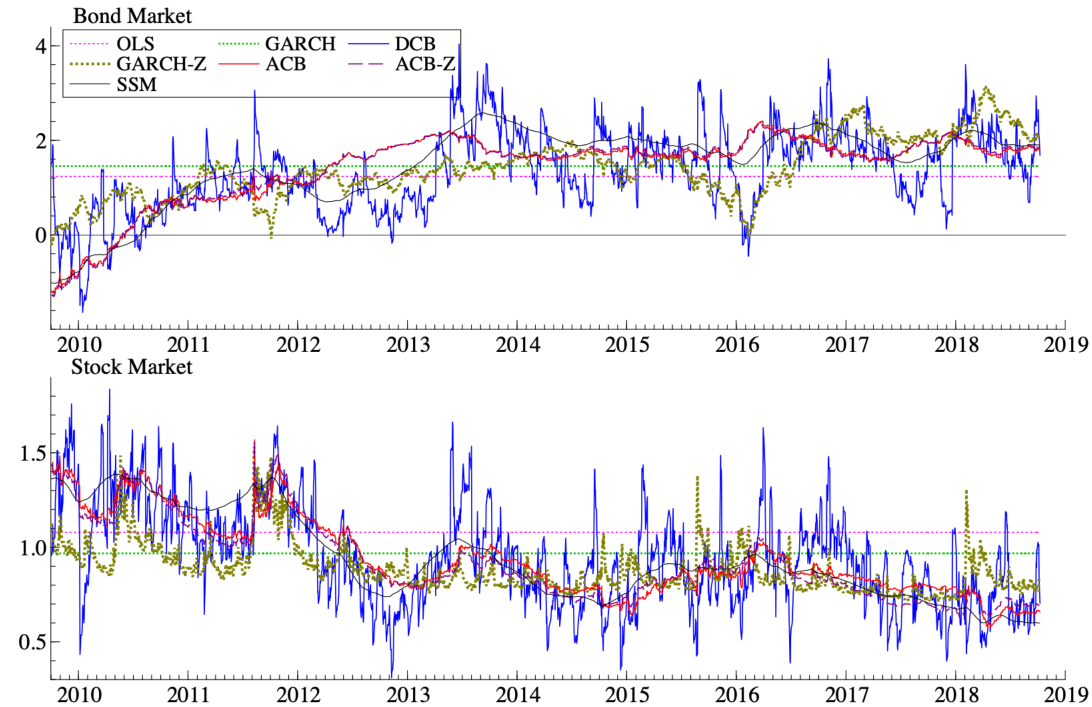

Figure 2: In-sample conditional betas for developed Europe

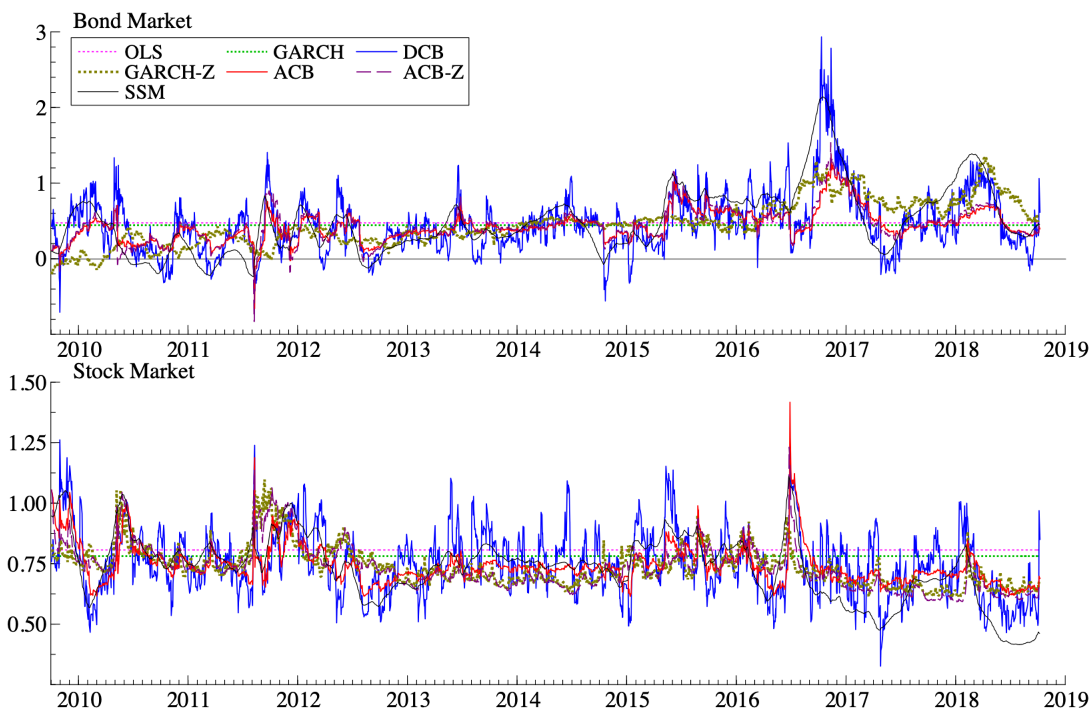

Figure 3: Out-of-sample conditional betas for the U.S.

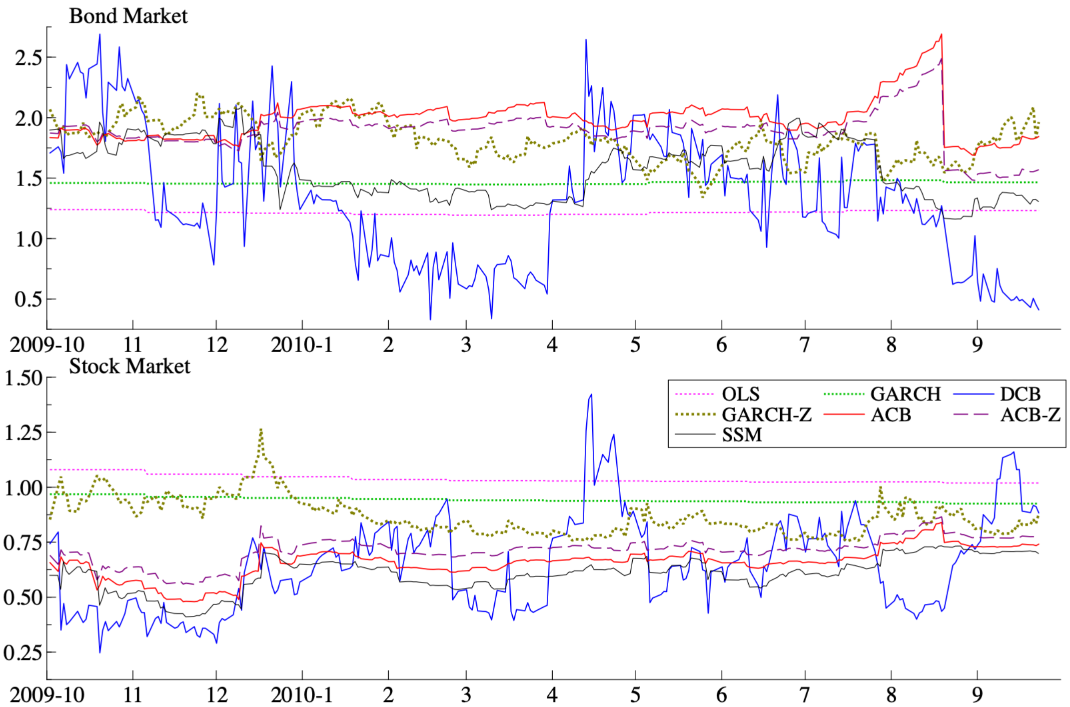

Figure 4: Out-of-sample conditional betas for developed Europe