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# Supplement to the Paper: Unit Root Test with High-Frequency Data[1]

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This paper provides a supplement to the paper "Unit Root Test with High-Frequency Data". It provides a detailed proof for some of the Lemmas and Remarks.

## 1. ASYMPTOTICS OF THE DF STATISTIC

The null hypothesis of a unit root is specified as

$$y_{i\Delta} = y_{(i-1)\Delta} + \sigma\sqrt{\Delta}\varepsilon_{i\Delta}, \tag{S1}$$

with initial value $y_0$, where $\sigma$ is a constant and $\varepsilon_{i\Delta} \overset{i.i.d}{\sim} \mathcal{N}(0,1)$. The alternative hypothesis is

$$y_{i\Delta} = \alpha_0 + \beta_0 y_{(i-1)\Delta} + \lambda_0 \varepsilon_{i\Delta}, \tag{S2}$$

where $\alpha_0 = \mu\left(1-e^{\theta\Delta}\right)$ with $\mu$ and $\theta$ being constant, $\beta_0 = e^{\theta\Delta}$ and $\lambda_0^2 = \frac{\sigma^2}{2\theta}\left(e^{2\theta\Delta}-1\right)$. The regression model is specified as follows:

$$y_{i\Delta} = \alpha + \beta y_{(i-1)\Delta} + v_{i\Delta}, \tag{S3}$$

where $v_{i\Delta}$ is the error term. The Dickey-Fuller statistic is

$$DF = \left(\hat{\beta}-1\right)\left[\frac{T\sum_{j=1}^{T} y_{(j-1)\Delta}^2 - \left(\sum_{j=1}^{T} y_{(j-1)\Delta}\right)^2}{\sum_{j=1}^{T}\left(y_{j\Delta}-\hat{\alpha}-\hat{\beta}y_{(j-1)\Delta}\right)^2}\right]^{1/2}, \tag{S4}$$

where $\hat{\alpha}$ and $\hat{\beta}$ represent the OLS estimates of $\alpha$ and $\beta$.

*Proof of Lemma 2.1.* $(a)$ Under model (S1),

$$y_{T\Delta} = \sigma\sqrt{\Delta}\sum_{j=1}^{T}\varepsilon_{j\Delta} + y_0 = \sigma N^{1/2}\left(T^{-1/2}\sum_{j=1}^{T}\varepsilon_{j\Delta}\right) + y_0 \Longrightarrow \sigma N^{1/2}\left(w_1 + \frac{y_0}{\sigma N^{1/2}}\right) \equiv \sigma N^{1/2}\Psi_1,$$

given $T^{-1/2}\sum_{j=1}^{T}\varepsilon_{j\Delta} \Longrightarrow w_1$.
$(b)$ The quantity

$$\begin{aligned} T^{-1}\sum_{j=1}^{T} y_{j\Delta} &= T^{-1}\sum_{j=1}^{T}\left[\sigma\sqrt{\Delta}\sum_{i=1}^{j}\varepsilon_{i\Delta} + y_0\right] \\ &= \sigma N^{1/2}\left[\frac{1}{T}\sum_{j=1}^{T}\left(T^{-1/2}\sum_{i=1}^{j}\varepsilon_{i\Delta}\right)\right] + T^{-1}\sum_{j=1}^{T} y_0 \\ &\Longrightarrow \sigma N^{1/2}\left(\int_0^1 w_s ds + \frac{y_0}{\sigma N^{1/2}}\right) \equiv \sigma N^{1/2}\Psi_2. \end{aligned}$$

$(c)$ The quantity

$$T^{-1}\sum_{j=1}^{T} y_{j\Delta}^2 = T^{-1}\sum_{j=1}^{T}\left[\sigma\sqrt{\Delta}\sum_{i=1}^{j}\varepsilon_{i\Delta} + y_0\right]^2$$

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$$
= \sigma^2 \Delta T \left[ \frac{1}{T} \sum_{j=1}^{T} \left( T^{-1/2} \sum_{i=1}^{j} \varepsilon_{i\Delta} \right)^2 \right] + y_0^2 + 2 y_0 \sigma \sqrt{\Delta} T^{1/2} \left[ \frac{1}{T} \sum_{j=1}^{T} \left( T^{-1/2} \sum_{i=1}^{j} \varepsilon_{i\Delta} \right) \right]
$$

$$
\Longrightarrow \sigma^2 N \left[ \int_0^1 w_s^2 ds + \frac{y_0^2}{\sigma^2 N} + 2 \frac{y_0}{\sigma N^{1/2}} \int_0^1 w_s ds \right] \equiv \sigma^2 N \Psi_3 .
$$

$(d)$ Note that by squaring equation (S1), subtracting $y_{(i-1)\Delta}^2$ from both sides, summing over $j = 1, ..., T$, re-organizing the equation, and multiplying by $T^{-1/2}$ we get

$$
\begin{aligned}
T^{-1/2} \sum_{j=1}^{T} y_{(j-1)\Delta} \varepsilon_{j\Delta} &= \frac{1}{2\sigma N^{1/2}} \left[ \sum_{j=1}^{T} \left( y_{j\Delta}^2 - y_{(j-1)\Delta}^2 \right) - \sigma^2 \Delta \sum_{j=1}^{T} \varepsilon_{j\Delta}^2 \right] \\
&= \frac{1}{2\sigma N^{1/2}} \left[ y_{T\Delta}^2 - y_0^2 - \sigma^2 N \left( T^{-1} \sum_{j=1}^{T} \varepsilon_{j\Delta}^2 \right) \right] \\
&\Longrightarrow \frac{1}{2} \sigma N^{1/2} \left( w_1^2 + 2 \frac{y_0}{\sigma N^{1/2}} w_1 - 1 \right) \equiv \sigma N^{1/2} \Psi_4
\end{aligned}
$$

since $y_{T\Delta}^2 \Longrightarrow \Psi_1$ and $T_n^{-1} \sum_{j=1}^{T_n} \varepsilon_j^2 \to 1$. □

*Proof of Lemma 2.2.* $(a)$ Use (S2) to obtain

$$
\begin{aligned}
y_{T\Delta} &= \alpha_0 \frac{1 - e^{T\theta\Delta}}{1 - e^{\theta\Delta}} + \lambda_0 \sum_{j=1}^{T} e^{(T-j)\theta\Delta} \varepsilon_{j\Delta} + e^{T\theta\Delta} y_0 \\
&= \alpha_0 \frac{1 - e^{c}}{1 - e^{\theta\Delta}} + \lambda_0 \sum_{j=1}^{T} e^{\frac{T-j}{T} c} \varepsilon_{j\Delta} + e^{c} y_0 \\
&= \mu \left( 1 - e^{c} \right) + \Delta^{1/2} T^{1/2} \frac{\lambda_0}{\Delta^{1/2}} \left( T^{-1/2} \sum_{j=1}^{T} e^{\frac{T-j}{T} c} \varepsilon_{j\Delta} \right) + e^{c} y_0 \\
&\Longrightarrow \sigma N^{1/2} \left[ \frac{\mu}{\sigma N^{1/2}} \left( 1 - e^{c} \right) + J_c(1) + e^{c} \frac{y_0}{\sigma N^{1/2}} \right] \equiv \sigma N^{1/2} \Xi_1
\end{aligned}
$$

since

$$
T^{-1/2} \sum_{j=1}^{T} e^{\frac{T-j}{T} c} \varepsilon_{j\Delta} \Longrightarrow J_c(1) = \int_0^1 e^{c(1-s)} dw_s
$$

from Lemma 1$(a)$ of Phillips (1987a), and

$$
\frac{\lambda_0^2}{\Delta} = \frac{\frac{\sigma^2}{2\theta} \left( e^{2\theta\Delta} - 1 \right)}{\Delta} \to \sigma^2 \text{ as } \Delta \to 0.
$$

$(b)$ We have

$$
\begin{aligned}
T^{-1} \sum_{i=1}^{T} y_{i\Delta} &= T^{-1} \sum_{i=1}^{T} \left[ \alpha_0 \frac{1 - e^{i\theta\Delta}}{1 - e^{\theta\Delta}} + \lambda_0 \sum_{j=1}^{i} e^{(i-j)\theta\Delta} \varepsilon_{j\Delta} + e^{i\theta\Delta} y_0 \right] \\
&= T^{-1} \frac{\alpha_0}{1 - e^{\theta\Delta}} \left( T - \sum_{i=1}^{T} e^{i\theta\Delta} \right) + T^{-1} \lambda_0 \sum_{i=1}^{T} \sum_{j=1}^{i} e^{(i-j)\theta\Delta} \varepsilon_{j\Delta} + T^{-1} y_0 \sum_{i=1}^{T} e^{i\theta\Delta} \\
&= \mu \left( 1 - \frac{\Delta}{N} e^{\theta\Delta} \frac{1 - e^{c}}{1 - e^{\theta\Delta}} \right) + N^{1/2} \frac{\lambda_0}{\Delta^{1/2}} \left( T^{-3/2} \sum_{i=1}^{T} \sum_{j=1}^{i} e^{(i-j)\theta\Delta} \varepsilon_{j\Delta} \right) + y_0 \frac{\Delta}{N} e^{\theta\Delta} \frac{1 - e^{c}}{1 - e^{\theta\Delta}} \\
&\Longrightarrow \sigma N^{1/2} \left[ \frac{\mu}{\sigma N^{1/2}} + \int_0^1 J_c(r) dr + \frac{y_0 - \mu}{\sigma N^{1/2}} \frac{e^{c} - 1}{c} \right] \equiv \sigma N^{1/2} \Xi_2
\end{aligned}
$$

since

$$
T^{-3/2} \sum_{i=1}^{T} \sum_{j=1}^{i} e^{(i-j)\theta\Delta} \varepsilon_{j\Delta} \Longrightarrow \int_0^1 J_c(r) dr
$$

using Lemma 1$(b)$ of Phillips (1987b), and

$$
\frac{\Delta}{N} e^{\theta\Delta} \frac{1 - e^{c}}{1 - e^{\theta\Delta}} = \frac{e^{\theta\Delta}}{N} \left( 1 - e^{c} \right) \frac{\Delta}{1 - e^{\theta\Delta}} \to \frac{e^{c} - 1}{c}.
$$

$(c)$ Consider

$$
\begin{aligned}
T^{-1}\sum_{i=1}^{T} y_{i\Delta}^2 &= T^{-1}\sum_{i=1}^{T}\left[\alpha_0\frac{1-e^{i\theta\Delta}}{1-e^{\theta\Delta}}+\lambda_0\sum_{j=1}^{i}e^{(i-j)\theta\Delta}\varepsilon_{j\Delta}+e^{i\theta\Delta}y_0\right]^2\\
&= T^{-1}\sum_{i=1}^{T}\left[\left(\alpha_0\frac{1-e^{i\theta\Delta}}{1-e^{\theta\Delta}}\right)^2+\left(\lambda_0\sum_{j=1}^{i}e^{(i-j)\theta\Delta}\varepsilon_{j\Delta}\right)^2+e^{2i\theta\Delta}y_0^2\right.\\
&\left.+2\alpha_0\frac{1-e^{i\theta\Delta}}{1-e^{\theta\Delta}}\lambda_0\sum_{j=1}^{i}e^{(i-j)\theta\Delta}\varepsilon_{j\Delta}+2\alpha_0\frac{1-e^{i\theta\Delta}}{1-e^{\theta\Delta}}e^{i\theta\Delta}y_0+2\lambda_0\sum_{j=1}^{i}e^{(i-j)\theta\Delta}\varepsilon_{j\Delta}e^{i\theta\Delta}y_0\right]\\
&= \mu^2+2\mu\left(y_0-\mu\right)T^{-1}\sum_{i=1}^{T}e^{i\theta\Delta}+\left(y_0-\mu\right)^2T^{-1}\sum_{i=1}^{T}e^{2i\theta\Delta}\\
&+\lambda_0^2T^{-1}\sum_{i=1}^{T}\left(\sum_{j=1}^{i}e^{(i-j)\theta\Delta}\varepsilon_{j\Delta}\right)^2+2\mu\lambda_0T^{-1}\sum_{i=1}^{T}\sum_{j=1}^{i}e^{(i-j)\theta\Delta}\varepsilon_{j\Delta}\\
&-2\mu\lambda_0T^{-1}\sum_{i=1}^{T}e^{i\theta\Delta}\sum_{j=1}^{i}e^{(i-j)\theta\Delta}\varepsilon_{j\Delta}+2\lambda_0y_0T^{-1}\sum_{i=1}^{T}e^{i\theta\Delta}\sum_{j=1}^{i}e^{(i-j)\theta\Delta}\varepsilon_{j\Delta}\\
&\Longrightarrow \sigma^2N\left[\frac{\mu^2}{\sigma^2N}+2\frac{\mu\left(y_0-\mu\right)}{\sigma^2N}\frac{e^c-1}{c}+\left(\frac{y_0-\mu}{\sigma N^{1/2}}\right)^2\frac{e^{2c}-1}{2c}\right.\\
&\left.+\int_0^1J_c(r)^2dr+2\frac{\mu}{\sigma N^{1/2}}\int_0^1J_c(r)\,dr+2\frac{y_0-\mu}{\sigma N^{1/2}}\int_0^1e^{cr}J_c(r)\,dr\right]\\
&\equiv \sigma^2N\Xi_3
\end{aligned}
$$

since

$$
\begin{aligned}
T^{-3/2}\sum_{i=1}^{T}\left(\sum_{j=1}^{i}e^{(i-j)\theta\Delta}\varepsilon_{j\Delta}\right)^2 &= T^{-3/2}\sum_{i=1}^{T}\left(\sum_{j=1}^{i}e^{\frac{i-j}{T}c}\varepsilon_{j\Delta}\right)^2\Longrightarrow\int_0^1J_c(r)\,dr,\\
T^{-2}\sum_{i=1}^{T}\left(\sum_{j=1}^{i}e^{(i-j)\theta\Delta}\varepsilon_{j\Delta}\right)^2 &= T^{-1}\sum_{i=1}^{T}\left(T^{-1/2}\sum_{j=1}^{i}e^{(i-j)\theta\Delta}\varepsilon_{j\Delta}\right)^2\Longrightarrow\int_0^1J_c(r)^2dr,\\
T^{-3/2}\sum_{i=1}^{T}e^{i\theta\Delta}\sum_{j=1}^{i}e^{(i-j)\theta\Delta}\varepsilon_{j\Delta} &= T^{-3/2}\sum_{i=1}^{T}e^{\frac{i}{T}c}\sum_{j=1}^{i}e^{\frac{i-j}{T}c}\varepsilon_{j\Delta}\Longrightarrow\int_0^1e^{cr}J_c(r)\,dr.
\end{aligned}
$$

$(d)$ By squaring (S2), subtracting $y_{(i-1)\Delta}^2$ from both sides, summing over $t=1,...,T$, re-organizing the equation, and multiplying by $T^{-1/2}$ we get

$$
\begin{aligned}
T^{-1/2}\sum_{i=1}^{T}y_{(i-1)\Delta}\varepsilon_{i\Delta} &= \frac{T^{-1/2}}{2e^{\theta\Delta}\lambda_0}\left[\sum_{i=1}^{T}\left(y_{i\Delta}^2-y_{(i-1)\Delta}^2\right)-T\alpha_0^2-\left(e^{2\theta\Delta}-1\right)\sum_{i=1}^{T}y_{(i-1)\Delta}^2\right.\\
&\left.-\lambda_0^2\sum_{i=1}^{T}\varepsilon_{i\Delta}^2-2\alpha_0\lambda_0\sum_{i=1}^{T}\varepsilon_{i\Delta}-2\alpha_0e^{\theta\Delta}\sum_{i=1}^{T}y_{(i-1)\Delta}\right]\\
&= \frac{1}{2e^{\theta\Delta}\sigma N^{1/2}}\left[y_{T\Delta}^2-y_0^2-T\alpha_0^2-N\frac{e^{2\theta\Delta}-1}{\Delta}\left(T^{-1}\sum_{i=1}^{T}y_{(i-1)\Delta}^2\right)\right.\\
&\left.-N\frac{\lambda_0^2}{\Delta}\left(T^{-1}\sum_{i=1}^{T}\varepsilon_{i\Delta}^2\right)-2\alpha_0N^{1/2}\frac{\lambda_0}{\Delta^{1/2}}\left(T^{-1/2}\sum_{i=1}^{T}\varepsilon_{i\Delta}\right)-2Ne^{\theta\Delta}\frac{\alpha_0}{\Delta}\left(T^{-1}\sum_{i=1}^{T}y_{(i-1)\Delta}\right)\right]\\
&\Longrightarrow \frac{\sigma N^{1/2}}{2}\left(\Xi_1^2-\frac{y_0^2}{\sigma^2N}-2c\Xi_3-1+2c\frac{\mu}{\sigma N^{1/2}}\Xi_2\right)\equiv\sigma N^{1/2}\Xi_4.
\end{aligned}
$$

□

*Proof of Remark 2.2.* The DF statistic can be rewritten as

$$DF = \left[\left(\hat{\beta}-\beta_0\right)+(\beta_0-1)\right]\left[\frac{T\sum_{i=1}^{T}y_{(i-1)\Delta}^2-\left(\sum_{i=1}^{T}y_{(i-1)\Delta}\right)^2}{\sum_{i=1}^{T}\left(y_{i\Delta}-\hat{\alpha}-\hat{\beta}y_{(i-1)\Delta}\right)^2}\right]^{1/2}$$

$$= t_{\hat{\beta}}+(\beta_0-1)\left[\frac{T\sum_{i=1}^{T}y_{(i-1)\Delta}^2-\left(\sum_{i=1}^{T}y_{(i-1)\Delta}\right)^2}{\sum_{i=1}^{T}\left(y_{i\Delta}-\hat{\alpha}-\hat{\beta}y_{(i-1)\Delta}\right)^2}\right]^{1/2}.$$

(1) When $\theta>0$, from Wang and Yu (2016), under the DGP of (S2 ) and the double asymptotic scheme,

$$t_{\hat{\beta}}\Longrightarrow\mathcal{N}(0,1),$$

$$\frac{e^{2\theta\Delta}-1}{e^{2\theta N}}\sum_{i=1}^{T}y_{(i-1)\Delta}^2\Longrightarrow(y_0-\mu+\sigma Z_1)^2,$$

$$\frac{e^{\theta\Delta}-1}{e^{\theta N}}\sum_{i=1}^{T}y_{i\Delta}\Longrightarrow\sigma Z_1+y_0-\mu,$$

$$\frac{1}{N}\sum_{i=1}^{T}\left(y_{i\Delta}-\hat{\alpha}-\hat{\beta}y_{(i-1)\Delta}\right)^2\ \to\ \sigma^2,$$

with $Z_1\sim\mathcal{N}(0,\frac{1}{2\theta})$. Therefore,

$$(\beta_0-1)\frac{\left[T\sum_{i=1}^{T}y_{(i-1)\Delta}^2-\left(\sum_{i=1}^{T}y_{(i-1)\Delta}\right)^2\right]^{1/2}}{\left[\sum_{i=1}^{T}\left(y_{i\Delta}-\hat{\alpha}-\hat{\beta}y_{(i-1)\Delta}\right)^2\right]^{1/2}}=\left(e^{\theta\Delta}-1\right)\left[\frac{T\frac{e^{2\theta N}}{e^{2\theta\Delta}-1}(y_0-\mu+\sigma Z_1)^2}{\sigma^2N}\right]^{1/2}[1+o_p(1)]$$

$$\sim e^{\theta N}\sqrt{\frac{\theta}{2}}\frac{1}{\sigma}|y_0-\mu+\sigma Z_1|\to+\infty.$$

It follows that

$$DF=(\beta_0-1)\frac{\left[T\sum_{i=1}^{T}y_{(i-1)\Delta}^2-\left(\sum_{i=1}^{T}y_{(i-1)\Delta}\right)^2\right]^{1/2}}{\left[\sum_{i=1}^{T}\left(y_{i\Delta}-\hat{\alpha}-\hat{\beta}y_{(i-1)\Delta}\right)^2\right]^{1/2}}[1+o_p(1)]\sim e^{\theta N}\sqrt{\frac{\theta}{2}}\frac{1}{\sigma}|y_0-\mu+\sigma Z_1|\to+\infty.$$

(2) Consider the case of $\theta<0$. From Wang and Yu (2016), assuming $E|\varepsilon_{1\Delta}|^{2+\delta}<\infty$ for some $\delta>0$, under the DGP of (S2) and the double asymptotic scheme,

$$t_{\hat{\beta}}\Longrightarrow\mathcal{N}(0,1),$$

$$\frac{1}{T}\sum_{i=1}^{T}y_{(i-1)\Delta}^2\Longrightarrow-\frac{1}{2\theta}+\frac{\mu^2}{\sigma^2},$$

$$\frac{1}{T}\sum_{i=1}^{T}y_{i\Delta}\Longrightarrow\frac{\mu}{\sigma},$$

$$\frac{1}{N}\sum_{i=1}^{T}\left(y_{i\Delta}-\hat{\alpha}-\hat{\beta}y_{(i-1)\Delta}\right)^2\ \to\ \sigma^2.$$

Therefore,

$$(\beta_0-1)\frac{\left[T\sum_{i=1}^{T}y_{(i-1)\Delta}^2-\left(\sum_{i=1}^{T}y_{(i-1)\Delta}\right)^2\right]^{1/2}}{\left[\sum_{i=1}^{T}\left(y_{i\Delta}-\hat{\alpha}-\hat{\beta}y_{(i-1)\Delta}\right)^2\right]^{1/2}}$$

$$=\frac{T}{\sigma N^{1/2}}\left(e^{\theta\Delta}-1\right)\left[\frac{1}{T}\sum_{i=1}^{T}y_{(i-1)\Delta}^2-\left(\frac{1}{T}\sum_{i=1}^{T}y_{i\Delta}\right)^2\right]^{1/2}[1+o_p(1)]$$

$$\sim N^{1/2}\frac{\theta}{\sigma}\left(-\frac{1}{2\theta}\right)^{1/2}\to-\infty.$$

It follows that when $\theta < 0$

$$DF = (\beta_0 - 1)\frac{\left[T\sum_{i=1}^{T} y_{(i-1)\Delta}^2 - \left(\sum_{i=1}^{T} y_{(i-1)\Delta}\right)^2\right]^{1/2}}{\left[\sum_{i=1}^{T}\left(y_{i\Delta} - \hat{\alpha} - \hat{\beta} y_{(i-1)\Delta}\right)^2\right]^{1/2}}\left[1 + o_p(1)\right] \sim N^{1/2}\frac{\theta}{\sigma}\left(-\frac{1}{2\theta}\right)^{1/2} \to -\infty.$$

□

## 2. ASYMPTOTICS FOR MODELS WITH JUMPS

The null hypothesis of a unit root with jumps is

$$y_{i\Delta} = \sum_{k=1}^{K}\phi_k I_{i\Delta}^k + y_{(i-1)\Delta} + \sigma\sqrt{\Delta}\varepsilon_{i\Delta}, \tag{S5}$$

and the alternative is

$$y_{i\Delta} = \alpha_0 + \sum_{k=1}^{K}\phi_k I_{i\Delta}^k + \beta_0 y_{(i-1)\Delta} + \lambda_0 \varepsilon_{i\Delta}, \tag{S6}$$

where $\sum_{k=1}^{K}\phi_k I_{i\Delta}^k$ is the jump component, defined in the main paper.

*Proof of Lemma 3.1.* The null hypothesis (S5) can be rewritten as

$$y_{i\Delta} = \sum_{j=1}^{i}\sum_{k=1}^{K}\phi_k I_{j\Delta}^k + \sigma\sqrt{\Delta}\sum_{j=1}^{i}\varepsilon_{j\Delta} + y_0. \tag{S7}$$

$(a)$ We have

$$y_{T\Delta} = \sum_{j=1}^{T}\sum_{k=1}^{K}\phi_k I_{j\Delta}^k + \sigma\sqrt{\Delta}\sum_{j=1}^{T}\varepsilon_{j\Delta} + y_0 \Longrightarrow \sigma N^{1/2}\left[\sum_{k=1}^{K}\frac{\phi_k}{\sigma N^{1/2}} + \Psi_1\right] \equiv \sigma N^{1/2}\tilde{\Psi}_1$$

since

$$\sum_{j=1}^{T}\sum_{k=1}^{K}\phi_k I_{j\Delta}^k = \sum_{k=1}^{K}\phi_k\sum_{j=1}^{T} I_{j\Delta}^k = \sum_{k=1}^{K}\phi_k$$

and from the proof of Lemma 2.1

$$\sigma\sqrt{\Delta}\sum_{j=1}^{T}\varepsilon_{j\Delta} + y_0 \Longrightarrow \sigma N^{1/2}\Psi_1.$$

$(b)$ The quantity

$$\begin{aligned} T^{-1}\sum_{i=1}^{T} y_{i\Delta} &= T^{-1}\sum_{i=1}^{T}\sum_{j=1}^{i}\sum_{k=1}^{K}\phi_k I_{j\Delta}^k + T^{-1}\sum_{i=1}^{T}\left(\sigma\sqrt{\Delta}\sum_{j=1}^{i}\varepsilon_{j\Delta} + y_0\right) \\ &\Longrightarrow \sigma N^{1/2}\left[\sum_{k=1}^{K}\frac{\phi_k}{\sigma N^{1/2}}(1 - r_k) + \Psi_2\right] \equiv \sigma N^{1/2}\tilde{\Psi}_2 \end{aligned}$$

since

$$\begin{aligned} T^{-1}\sum_{i=1}^{T}\sum_{j=1}^{i}\sum_{k=1}^{K}\phi_k I_{j\Delta}^k &= T^{-1}\sum_{k=1}^{K}\phi_k\sum_{i=1}^{T}\sum_{j=1}^{i} I_{j\Delta}^k \\ &= T^{-1}\sum_{k=1}^{K}\phi_k\left[I_{1\Delta}^k + (I_{1\Delta}^k + I_{2\Delta}^k) + \cdots + (I_{1\Delta}^k + I_{2\Delta}^k + \cdots + I_{T\Delta}^k)\right] \\ &= T^{-1}\sum_{k=1}^{K}\phi_k(T - \tau_k) = \sum_{k=1}^{K}\phi_k(1 - r_k) \end{aligned}$$

and from the proof of Lemma 2.1,

$$T^{-1}\sum_{i=1}^{T}\left(\sigma\sqrt{\Delta}\sum_{j=1}^{i}\varepsilon_{j\Delta} + y_0\right) \Longrightarrow \sigma N^{1/2}\Psi_2.$$

$(c)$ The quantity

$$T^{-1}\sum_{i=1}^{T}y_{i\Delta}^{2}=T^{-1}\sum_{i=1}^{T}\left[\sum_{j=1}^{i}\sum_{k=1}^{K}\phi_k I_{j\Delta}^{k}+\sigma\sqrt{\Delta}\sum_{j=1}^{i}\varepsilon_{j\Delta}+y_0\right]^2$$
$$=T^{-1}\sum_{i=1}^{T}\left(\sum_{j=1}^{i}\sum_{k=1}^{K}\phi_k I_{j\Delta}^{k}\right)^2+T^{-1}\sum_{i=1}^{T}\left(\sigma\sqrt{\Delta}\sum_{j=1}^{i}\varepsilon_{j\Delta}+y_0\right)^2$$
$$+2T^{-1}\sum_{i=1}^{T}\sum_{j=1}^{i}\sum_{k=1}^{K}\phi_k I_{j\Delta}^{k}\left(\sigma\sqrt{\Delta}\sum_{j=1}^{i}\varepsilon_{j\Delta}+y_0\right).$$

The first term

$$T^{-1}\sum_{i=1}^{T}\left(\sum_{j=1}^{i}\sum_{k=1}^{K}\phi_k I_{j\Delta}^{k}\right)^2=T^{-1}\sum_{i=1}^{T}\left(\sum_{k=1}^{K}\phi_k\sum_{j=1}^{i}I_{j\Delta}^{k}\right)^2$$
$$=T^{-1}\left[\sum_{k=1}^{K-1}(\tau_{k+1}-\tau_k)\left(\sum_{j=1}^{k}\phi_j\right)^2+(T-\tau_K)\left(\sum_{j=1}^{K}\phi_j\right)^2\right]$$
$$=\begin{cases}(1-r_1)\,\phi_1^2 & \text{if } K=1\\ \sum_{k=1}^{K-1}(r_{k+1}-r_k)\left(\sum_{j=1}^{k}\phi_j\right)^2+(1-r_K)\left(\sum_{j=1}^{K}\phi_j\right)^2 & \text{if } K>1.\end{cases}$$

The second term, from the proof of Lemma 2.1,

$$T^{-1}\sum_{j=1}^{T}\left[\sigma\sqrt{\Delta}\sum_{i=1}^{j}\varepsilon_{i\Delta}+y_0\right]^2\Longrightarrow\sigma^2 N\Psi_3.$$

The third term

$$2T^{-1}\sum_{i=1}^{T}\sum_{j=1}^{i}\sum_{k=1}^{K}\phi_k I_{j\Delta}^{k}\left(\sigma\sqrt{\Delta}\sum_{j=1}^{i}\varepsilon_{j\Delta}+y_0\right)$$
$$=\ 2\sigma N^{1/2}\left(T^{-3/2}\sum_{k=1}^{K}\phi_k\sum_{i=1}^{T}\sum_{j=1}^{i}I_{j\Delta}^{k}\sum_{j=1}^{i}\varepsilon_{j\Delta}\right)+2y_0\left(T^{-1}\sum_{i=1}^{T}\sum_{j=1}^{i}\sum_{k=1}^{K}\phi_k I_{j\Delta}^{k}\right)$$
$$\Longrightarrow 2\sigma N^{1/2}\sum_{k=1}^{K}\phi_k\int_{r_k}^{1}w_s ds+2y_0\sum_{k=1}^{K}\phi_k(1-r_k)$$

since

$$T^{-3/2}\sum_{i=1}^{T}\left(\sum_{k=1}^{K}\phi_k\sum_{j=1}^{i}I_{j\Delta}^{k}\right)\left(\sum_{j=1}^{i}\varepsilon_{j\Delta}\right)$$
$$=\ T^{-3/2}\sum_{k=1}^{K}\phi_k\sum_{i=1}^{T}\left(\sum_{j=1}^{i}I_{j\Delta}^{k}\sum_{j=1}^{i}\varepsilon_{j\Delta}\right)$$
$$=\ T_n^{-3/2}\sum_{k=1}^{K}\phi_k\left[(T-\tau_k)\sum_{i=1}^{\tau_k}\varepsilon_{i\Delta}+\sum_{i=\tau_k+1}^{T}(T-i+1)\,\varepsilon_{i\Delta}\right]$$
$$=\ \sum_{k=1}^{K}\phi_k\left[\frac{T-\tau_k}{T}\left(T^{-1/2}\sum_{i=1}^{\tau_k}\varepsilon_{i\Delta}\right)+T_n^{-3/2}\sum_{i=\tau_k+1}^{T_n}(T-i)\,\varepsilon_{i\Delta}\right]$$
$$=\ \sum_{k=1}^{K}\phi_k\left[\frac{T-\tau_k}{T}\left(T^{-1/2}\sum_{i=1}^{\tau_k}\varepsilon_{i\Delta}\right)+T^{-1/2}\sum_{i=\tau_k+1}^{T}\varepsilon_{i\Delta}-T^{-3/2}\sum_{i=\tau_k+1}^{T}i\varepsilon_{i\Delta}\right]$$
$$\Rightarrow\sum_{k=1}^{K}\phi_k\left[(1-r_k)\,w_{r_k}+\left(w_1-w_{r_k}\right)-\left(w_1-r_k w_{r_k}-\int_{r_k}^{1}w_s ds\right)\right]$$
$$=\ \sum_{k=1}^{K}\phi_k\int_{r_k}^{1}w_s ds.$$

Therefore,

$$T^{-1}\sum_{i=1}^{T}y_{i\Delta}^{2}\Rightarrow\sigma^{2}N\left[\Psi_{3}+\Delta_{1}+(1-r_{K})\left(\sum_{j=1}^{K}\frac{\phi_{j}}{\sigma N^{1/2}}\right)^{2}\right.$$
$$\left.+2\sum_{k=1}^{K}\frac{\phi_{k}}{\sigma N^{1/2}}\int_{r_{k}}^{1}w_{s}ds+2\frac{y_{0}}{\sigma N^{1/2}}\sum_{k=1}^{K}\frac{\phi_{k}}{\sigma N^{1/2}}(1-r_{k})\right]\equiv\sigma^{2}N\tilde{\Psi}_{3}$$

with

$$\Delta_{1}=\begin{cases}(1-r_{1})\frac{\phi_{1}^{2}}{\sigma^{2}N} & \text{if } K=1\\ \sum_{k=1}^{K-1}(r_{k+1}-r_{k})\left(\sum_{j=1}^{k}\frac{\phi_{j}}{\sigma N^{1/2}}\right)^{2}+(1-r_{K})\left(\sum_{j=1}^{K}\frac{\phi_{j}}{\sigma N^{1/2}}\right)^{2} & \text{if } K>1.\end{cases}$$

$(d)$ By squaring (S5), subtracting $y_{(i-1)\Delta}^{2}$ from both sides, summing over $i=1,...,T$, re-organizing the equation, and multiplying by $T^{-1/2}$, we get

$$T^{-1/2}\sum_{i=1}^{T}y_{(i-1)\Delta}\varepsilon_{i\Delta}=\frac{T^{-1/2}}{2\sigma\sqrt{\Delta}}\left[y_{T\Delta}^{2}-y_{0}^{2}-\sigma^{2}\Delta\sum_{i=1}^{T}\varepsilon_{i\Delta}^{2}-\sum_{i=1}^{T}\left(\sum_{k=1}^{K}\phi_{k}I_{i\Delta}^{k}\right)^{2}\right.$$
$$\left.-2\sum_{i=1}^{T}\sum_{k=1}^{K}\phi_{k}I_{i\Delta}^{k}\left(y_{(i-1)\Delta}+\sigma\sqrt{\Delta}\varepsilon_{(i-1)\Delta}\right)\right].$$

We have $y_{T\Delta}\Longrightarrow\tilde{\Psi}_{1},\sigma^{2}\Delta\sum_{i=1}^{T}\varepsilon_{i\Delta}^{2}\to\sigma^{2}N$, $\sum_{i=1}^{T}\left(\sum_{k=1}^{K}\phi_{k}I_{i\Delta}^{k}\right)^{2}=\sum_{k=1}^{K}\phi_{k}^{2}$ and

$$\begin{aligned}&2\sum_{i=1}^{T}\left(\sum_{k=1}^{K}\phi_{k}I_{i\Delta}^{k}\right)\left(y_{(i-1)\Delta}+\sigma\sqrt{\Delta}\varepsilon_{(i-1)\Delta}\right)\\&=2\sum_{k=1}^{K}\phi_{k}\sum_{i=1}^{T}I_{i\Delta}^{k}y_{(i-1)\Delta}+\sigma\sqrt{\Delta}\sum_{k=1}^{K}\phi_{k}\sum_{i=1}^{T}I_{i\Delta}^{k}\varepsilon_{(i-1)\Delta}\\&=2\sum_{k=1}^{K}\phi_{k}y_{(\tau_{k}-1)\Delta}+\sigma\sqrt{\Delta}\sum_{k=1}^{K}\phi_{k}\varepsilon_{(\tau_{k}-1)\Delta}\Longrightarrow2\sigma N^{1/2}\sum_{k=1}^{K}\phi_{k}\left(w_{r_{k}}+\frac{y_{0}}{\sigma N^{1/2}}\right)\end{aligned}$$

since $\sigma\sqrt{\Delta}\sum_{k=1}^{K}\phi_{k}\varepsilon_{(\tau_{k}-1)\Delta}\to0$ and

$$y_{(\tau_{k-1})\Delta}=\sigma\sqrt{\Delta}\sum_{j=1}^{\tau_{k-1}}\varepsilon_{j}+y_{0}+\sum_{k=1}^{K}\phi_{k}\sum_{j=1}^{\tau_{k-1}}I_{j}^{k}\Longrightarrow N^{1/2}\left(w_{r_{k}}+\frac{y_{0}}{\sigma N^{1/2}}+\Delta_{2}\right),$$

where $\Delta_{2}=0$ if $K=0$ and $\Delta_{2}=\sum_{j=1}^{k-1}\frac{\phi_{j}}{\sigma N^{1/2}}$ if $K>1$. Therefore,

$$\begin{aligned}T^{-1/2}\sum_{i=1}^{T}y_{(i-1)\Delta}\varepsilon_{i\Delta}&\Longrightarrow\frac{\sigma N^{1/2}}{2}\left[\tilde{\Psi}_{1}^{2}-\frac{y_{0}^{2}}{\sigma^{2}N}-1-\sum_{k=1}^{K}\frac{\phi_{k}^{2}}{\sigma^{2}N}-2\sum_{k=1}^{K}\frac{\phi_{k}}{\sigma N^{1/2}}\left(w_{r_{k}}+\frac{y_{0}}{\sigma N^{1/2}}+\Delta_{2}\right)\right]\\&\equiv\sigma N^{1/2}\tilde{\Psi}_{4}.\end{aligned}$$

□

*Proof of Lemma 3.2.* $(a)$ The alternative model can be rewritten as

$$y_{i\Delta}=\alpha_{0}\frac{1-e^{i\theta\Delta}}{1-e^{\theta\Delta}}+\sum_{j=1}^{i}e^{(i-j)\theta\Delta}\left(\lambda_{0}\varepsilon_{j\Delta}+\sum_{k=1}^{K}\phi_{k}I_{j\Delta}^{k}\right)+e^{i\theta\Delta}y_{0}. \tag{2.1}$$

It follows that

$$y_{T\Delta}=\alpha_{0}\frac{1-e^{T\theta\Delta}}{1-e^{\theta\Delta}}+\lambda_{0}\sum_{j=1}^{T}e^{(T-j)\theta\Delta}\varepsilon_{j\Delta}+e^{T\theta\Delta}y_{0}+\sum_{j=1}^{T}e^{(T-j)\theta\Delta}\sum_{k=1}^{K}\phi_{k}I_{j\Delta}^{k}.$$

From the proof of Lemma 2.2,

$$\alpha_{0}\frac{1-e^{T\theta\Delta}}{1-e^{\theta\Delta}}+\lambda_{0}\sum_{j=1}^{T}e^{(T-j)\theta\Delta}\varepsilon_{j\Delta}+e^{T\theta\Delta}y_{0}\Longrightarrow\sigma N^{1/2}\Xi_{1}.$$

The last term

$$\sum_{j=1}^{T} e^{(T-j)\theta\Delta} \sum_{k=1}^{K} \phi_k I_{j\Delta}^k = \sum_{k=1}^{K} \phi_k \sum_{j=1}^{T} e^{\frac{T-j}{T}c} I_{j\Delta}^k = \sum_{k=1}^{K} \phi_k e^{(1-r_k)c}.$$

Therefore,

$$y_{T\Delta} \Longrightarrow \sigma N^{1/2} \left[ \Xi_1 + \sum_{k=1}^{K} \frac{\phi_k}{\sigma N^{1/2}} e^{(1-r_k)c} \right] \equiv \sigma N^{1/2} \tilde{\Xi}_1.$$

(*b*) The quantity

$$\begin{aligned} T^{-1} \sum_{i=1}^{T} y_{i\Delta} &= T^{-1} \sum_{i=1}^{T} \left[ \alpha_0 \frac{1-e^{i\theta\Delta}}{1-e^{\theta\Delta}} + \lambda_0 \sum_{j=1}^{i} e^{(T-j)\theta\Delta} \varepsilon_{j\Delta} + e^{i\theta\Delta} y_0 + \sum_{j=1}^{i} e^{(i-j)\theta\Delta} \sum_{k=1}^{K} \phi_k I_{j\Delta}^k \right] \\ &= T^{-1} \sum_{i=1}^{T} \left[ \alpha_0 \frac{1-e^{i\theta\Delta}}{1-e^{\theta\Delta}} + \lambda_0 \sum_{j=1}^{i} e^{(T-j)\theta\Delta} \varepsilon_{j\Delta} + e^{i\theta\Delta} y_0 \right] + T^{-1} \sum_{i=1}^{T} \sum_{j=1}^{i} e^{(i-j)\theta\Delta} \sum_{k=1}^{K} \phi_k I_{j\Delta}^k. \end{aligned}$$

From the proof of Lemma 2.2,

$$T^{-1} \sum_{i=1}^{T} \left[ \alpha_0 \frac{1-e^{i\theta\Delta}}{1-e^{\theta\Delta}} + \lambda_0 \sum_{j=1}^{i} e^{(T-j)\theta\Delta} \varepsilon_{j\Delta} + e^{i\theta\Delta} y_0 \right] \Longrightarrow \sigma N^{1/2} \Xi_2.$$

Furthermore,

$$\begin{aligned} T^{-1} \sum_{i=1}^{T} \sum_{j=1}^{i} e^{(i-j)\theta\Delta} \sum_{k=1}^{K} \phi_k I_{j\Delta}^k &= T^{-1} \sum_{k=1}^{K} \phi_k \sum_{i=1}^{T} \sum_{j=1}^{i} e^{(i-j)\theta\Delta} I_{j\Delta}^k \\ &= T^{-1} \sum_{k=1}^{K} \phi_k \sum_{i=1}^{T} \frac{1-e^{(T-i+1)\theta\Delta}}{1-e^{\theta\Delta}} I_{i\Delta}^k \\ &= T^{-1} \sum_{k=1}^{K} \phi_k \frac{1-e^{(T-\tau_k+1)\theta\Delta}}{1-e^{\theta\Delta}} \\ &\to \frac{1}{c} \sum_{k=1}^{K} \phi_k \left[ e^{(1-r_k)c} - 1 \right]. \end{aligned}$$

Therefore,

$$T^{-1} \sum_{i=1}^{T} y_{i\Delta} \Longrightarrow \sigma N^{1/2} \left\{ \Xi_2 + \frac{1}{c} \sum_{k=1}^{K} \frac{\phi_k}{\sigma N^{1/2}} \left[ e^{(1-r_k)c} - 1 \right] \right\} \equiv \sigma N^{1/2} \tilde{\Xi}_2.$$

(*c*) The quantity

$$\begin{aligned} T^{-1} \sum_{i=1}^{T} y_{i\Delta}^2 =& T^{-1} \sum_{i=1}^{T} \left[ \alpha_0 \frac{1-e^{i\theta\Delta}}{1-e^{\theta\Delta}} + \lambda_0 \sum_{j=1}^{i} e^{(i-j)\theta\Delta} \varepsilon_{j\Delta} + e^{i\theta\Delta} y_0 + \sum_{j=1}^{i} e^{(i-j)\theta\Delta} \sum_{k=1}^{K} \phi_k I_{j\Delta}^k \right]^2 \\ &= T^{-1} \sum_{i=1}^{T} \left( \alpha_0 \frac{1-e^{i\theta\Delta}}{1-e^{\theta\Delta}} + \lambda_0 \sum_{j=1}^{i} e^{(i-j)\theta\Delta} \varepsilon_{j\Delta} + e^{i\theta\Delta} y_0 \right)^2 + T^{-1} \sum_{i=1}^{T} \left( \sum_{j=1}^{i} e^{(i-j)\theta\Delta} \sum_{k=1}^{K} \phi_k I_{j\Delta}^k \right)^2 \\ &+ 2T^{-1} \sum_{i=1}^{T} \left( \alpha_0 \frac{1-e^{i\theta\Delta}}{1-e^{\theta\Delta}} \right) \left( \sum_{j=1}^{i} e^{(i-j)\theta\Delta} \sum_{k=1}^{K} \phi_k I_{j\Delta}^k \right) \\ &+ 2T^{-1} \sum_{i=1}^{T} \lambda_0 \sum_{j=1}^{i} e^{(i-j)\theta\Delta} \varepsilon_{j\Delta} \left( \sum_{j=1}^{i} e^{(i-j)\theta\Delta} \sum_{k=1}^{K} \phi_k I_{j\Delta}^k \right) \\ &+ 2T^{-1} \sum_{i=1}^{T} e^{i\theta\Delta} y_0 \left( \sum_{j=1}^{i} e^{(i-j)\theta\Delta} \sum_{k=1}^{K} \phi_k I_{j\Delta}^k \right). \end{aligned}$$

From the proof of Lemma 2.2,

$$T^{-1} \sum_{i=1}^{T} \left( \alpha_0 \frac{1-e^{i\theta\Delta}}{1-e^{\theta\Delta}} + \lambda_0 \sum_{j=1}^{i} e^{(i-j)\theta\Delta} \varepsilon_{j\Delta} + e^{i\theta\Delta} y_0 \right)^2 \Longrightarrow \sigma^2 N \Xi_3.$$

The second term

$$T^{-1}\sum_{i=1}^{T}\left(\sum_{j=1}^{i}e^{(i-j)\theta\Delta}\sum_{k=1}^{K}\phi_k I_{j\Delta}^{k}\right)^2 = T^{-1}\sum_{i=1}^{T}\left(\sum_{k=1}^{K}\phi_k\sum_{j=1}^{i}e^{(i-j)\theta\Delta}I_{j\Delta}^{k}\right)^2.$$

If $K=1$, we have

$$T^{-1}\sum_{i=1}^{T}\left(\sum_{k=1}^{K}\phi_k\sum_{j=1}^{i}e^{(i-j)\theta\Delta}I_{j\Delta}^{k}\right)^2 = T^{-1}\sum_{i=\tau_1}^{T}\phi_1^2 e^{2(i-\tau_1)\theta\Delta} = \frac{\Delta}{N}\phi_1^2\frac{e^{2c(1-r_1)}-1}{e^{2\theta\Delta}-1} \to \phi_1^2\frac{1}{2c}\left[e^{2c(1-r_1)}-1\right].$$

If $K>1$,

$$\begin{aligned}
&T^{-1}\sum_{i=1}^{T}\left(\sum_{k=1}^{K}\phi_k\sum_{j=1}^{i}e^{(i-j)\theta\Delta}I_{j\Delta}^{k}\right)^2\\
&= T^{-1}\sum_{k=1}^{K-1}\sum_{i=\tau_k}^{\tau_{k+1}-1}\left(\sum_{j=1}^{k}\phi_j e^{\left(i-\tau_j\right)\theta\Delta}\right)^2 + T^{-1}\sum_{i=\tau_K}^{T}\left(\sum_{j=1}^{K}\phi_j e^{\left(i-\tau_j\right)\theta\Delta}\right)^2\\
&= T^{-1}\sum_{k=1}^{K-1}\sum_{i=\tau_k}^{\tau_{k+1}-1}e^{2i\theta\Delta}\left(\sum_{j=1}^{k}\phi_j e^{-\tau_j\theta\Delta}\right)^2 + T^{-1}\sum_{i=\tau_K}^{T}e^{2i\theta\Delta}\left(\sum_{j=1}^{K}\phi_j e^{-\tau_j\theta\Delta}\right)^2\\
&= T^{-1}\sum_{k=1}^{K-1}\left(\sum_{j=1}^{k}\phi_j e^{-\tau_j\theta\Delta}\right)^2\left(\sum_{i=\tau_k}^{\tau_{k+1}-1}e^{2i\theta\Delta}\right) + T^{-1}\left(\sum_{j=1}^{K}\phi_j e^{-\tau_j\theta\Delta}\right)^2\sum_{i=\tau_K}^{T}e^{2i\theta\Delta}\\
&= T^{-1}\sum_{k=1}^{K-1}\left(\sum_{j=1}^{k}\phi_j e^{-\tau_j\theta\Delta}\right)^2\left[e^{2r_k\theta}\frac{1-e^{2\theta(r_{k+1}-r_k)}}{1-e^{2\theta\Delta}}\right] + T^{-1}\left(\sum_{j=1}^{K}\phi_j e^{-\tau_j\theta\Delta}\right)^2\left[e^{2r_K\theta}\frac{1-e^{2\theta(1-r_K)}}{1-e^{2\theta\Delta}}\right]\\
&\to \frac{1}{2c}\left[\sum_{k=1}^{K-1}\left(\sum_{j=1}^{k}\phi_j e^{-r_j\theta}\right)^2 e^{2r_k\theta}\left[e^{2\theta(r_{k+1}-r_k)}-1\right] + \left(\sum_{j=1}^{K}\phi_j e^{-r_j\theta}\right)^2 e^{2r_K\theta}\left[e^{2\theta(1-r_K)}-1\right]\right].
\end{aligned}$$

The third term

$$\begin{aligned}
&2T^{-1}\sum_{i=1}^{T}\left(\alpha_0\frac{1-e^{i\theta\Delta}}{1-e^{\theta\Delta}}\right)\left(\sum_{j=1}^{i}e^{(i-j)\theta\Delta}\sum_{k=1}^{K}\phi_k I_{j\Delta}^{k}\right)\\
&= 2\frac{\alpha_0}{1-e^{\theta\Delta}}T^{-1}\sum_{i=1}^{T}\left(\sum_{j=1}^{i}e^{(i-j)\theta\Delta}\sum_{k=1}^{K}\phi_k I_{j\Delta}^{k}\right) - 2\frac{\alpha_0}{1-e^{\theta\Delta}}T^{-1}\sum_{i=1}^{T}e^{i\theta\Delta}\left(\sum_{j=1}^{i}e^{(i-j)\theta\Delta}\sum_{k=1}^{K}\phi_k I_{j\Delta}^{k}\right)\\
&= 2\mu T^{-1}\sum_{k=1}^{K}\phi_k\sum_{i=1}^{T}\sum_{j=1}^{i}e^{(i-j)\theta\Delta}I_{j\Delta}^{k} - 2\mu T^{-1}\sum_{k=1}^{K}\phi_k\sum_{i=1}^{T}\sum_{j=1}^{i}e^{i\theta\Delta}e^{(i-j)\theta\Delta}I_{j\Delta}^{k}\\
&= 2\mu T^{-1}\sum_{k=1}^{K}\phi_k e^{-\tau_k\theta\Delta}\sum_{i=\tau_k}^{T}e^{i\theta\Delta} - 2\mu T^{-1}\sum_{k=1}^{K}\phi_k e^{-\tau_k\theta\Delta}\sum_{i=\tau_k}^{T}e^{2i\theta\Delta}\\
&= 2\mu T^{-1}\sum_{k=1}^{K}\phi_k e^{-\tau_k\theta\Delta}\left[\sum_{i=\tau_k}^{T}e^{i\theta\Delta} - \sum_{i=\tau_k}^{T}e^{2i\theta\Delta}\right]\\
&= \frac{\mu}{c}\sum_{k=1}^{K}\phi_k\left[2e^{c(1-r_k)}-2-e^{c(2-r_k)}+e^{r_k c}\right].
\end{aligned}$$

The fourth term

$$\begin{aligned}
&2T^{-1}\sum_{i=1}^{T}\lambda_0\left(\sum_{j=1}^{i}e^{(i-j)\theta\Delta}\varepsilon_{j\Delta}\right)\left(\sum_{j=1}^{i}e^{(i-j)\theta\Delta}\sum_{k=1}^{K}\phi_k I_{j\Delta}^{k}\right)\\
&= 2\lambda_0 T^{-1}\sum_{k=1}^{K}\phi_k\sum_{i=1}^{T}\left(\sum_{j=1}^{i}e^{(i-j)\theta\Delta}\varepsilon_{j\Delta}\right)\left(\sum_{j=1}^{i}e^{(i-j)\theta\Delta}I_{j\Delta}^{k}\right)\\
&= 2\lambda_0 T^{1/2}\sum_{k=1}^{K}\phi_k\left[\frac{1}{T}\sum_{i=\tau_k}^{T}e^{(i-\tau_k)\theta\Delta}\left(T^{-1/2}\sum_{j=1}^{i}e^{(i-j)\theta\Delta}\varepsilon_{j\Delta}\right)\right]\\
&\Longrightarrow 2\sigma N^{1/2}\sum_{k=1}^{K}\phi_k\int_{r_k}^{1}e^{c(r-r_k)}J_c(r)\,dr.
\end{aligned}$$

The fifth term

$$2T^{-1}\sum_{i=1}^{T}e^{i\theta\Delta}y_0\left(\sum_{j=1}^{i}e^{(i-j)\theta\Delta}\sum_{k=1}^{K}\phi_k I_{j\Delta}^{k}\right)=2y_0T^{-1}\sum_{k=1}^{K}\phi_k\sum_{i=1}^{T}\left(\sum_{j=1}^{i}e^{i\theta\Delta}e^{(i-j)\theta\Delta}I_{j\Delta}^{k}\right)$$
$$=2y_0T^{-1}\sum_{k=1}^{K}\phi_k\sum_{i=\tau_k}^{T}e^{(2i-\tau_k)\theta\Delta}$$
$$=2y_0\sum_{k=1}^{K}\phi_k\left(e^{-\tau_k\theta\Delta}T^{-1}\sum_{i=\tau_k}^{T}e^{2i\theta\Delta}\right)$$
$$=\frac{y_0}{c}\sum_{k=1}^{K}\phi_k e^{r_k c}\left[e^{2c(1-r_k)}-1\right].$$

Therefore,

$$T^{-1}\sum_{i=1}^{T}y_{(i-1)\Delta}^{2}\Longrightarrow\sigma^2N\Xi_3+\frac{1}{2c}\sum_{k=1}^{K}\phi_k^2\left[e^{2(1-r_k)c}-1\right]+\frac{\mu}{c}\sum_{k=1}^{K}\phi_k\left[2e^{c(1-r_k)}-2-e^{c(2-r_k)}+e^{r_kc}\right]$$
$$+2\sigma N^{1/2}\sum_{k=1}^{K}\phi_k\int_{r_k}^{1}e^{c(r-r_k)}J_c(r)\,dr+\frac{y_0}{c}\sum_{k=1}^{K}\phi_k e^{r_kc}\left[e^{2c(1-r_k)}-1\right]$$
$$=\sigma^2N\left\{\Xi_3+\Delta_3+\frac{1}{c}\frac{\mu}{\sigma N^{1/2}}\sum_{k=1}^{K}\frac{\phi_k}{\sigma N^{1/2}}\left[2e^{c(1-r_k)}-2-e^{c(2-r_k)}+e^{r_kc}\right]\right.$$
$$\left.+2\sum_{k=1}^{K}\frac{\phi_k}{\sigma N^{1/2}}\int_{r_k}^{1}e^{c(r-r_k)}J_c(r)\,dr+\frac{1}{c}\frac{y_0}{\sigma N^{1/2}}\sum_{k=1}^{K}\frac{\phi_k}{\sigma N^{1/2}}e^{r_kc}\left[e^{2c(1-r_k)}-1\right]\right\}$$
$$\equiv\sigma^2N\tilde{\Xi}_3,$$

where

$$\Delta_3=\begin{cases}\frac{\phi_1^2}{\sigma^2N}\frac{1}{2c}\left[e^{2c(1-r_1)}-1\right] & \text{if } K=1\\ \frac{1}{2c}\left[\sum_{k=1}^{K-1}\left(\sum_{j=1}^{k}\frac{\phi_j}{\sigma N^{1/2}}e^{-r_j\theta}\right)^2e^{2r_k\theta}\left[e^{2\theta(r_{k+1}-r_k)}-1\right]+\left(\sum_{j=1}^{K}\frac{\phi_j}{\sigma N^{1/2}}e^{-r_j\theta}\right)^2e^{2r_K\theta}\left[e^{2\theta(1-r_K)}-1\right]\right] & \text{if } K>1\end{cases}.$$

$(d)$ By squaring (S6), subtracting $y_{(i-1)\Delta}^2$ from both sides, summing over $i=1,\dots,T$, re-organizing the equation, and multiplying $T^{-1}$ such that

$$T^{-1/2}\sum_{i=1}^{T}y_{(i-1)\Delta}\varepsilon_{i\Delta}$$
$$=\frac{T^{-1/2}}{2e^{\theta\Delta}\lambda_0}\left[\sum_{i=1}^{T}\left(y_{i\Delta}^2-y_{(i-1)\Delta}^2\right)-T\alpha_0^2-\left(e^{2\theta\Delta}-1\right)\sum_{i=1}^{T}y_{(i-1)\Delta}^2-\lambda_0^2\sum_{i=1}^{T}\varepsilon_{i\Delta}^2-2\alpha_0e^{\theta\Delta}\sum_{i=1}^{T}y_{(i-1)\Delta}\right.$$
$$\left.-2\alpha_0\lambda_0\sum_{i=1}^{T}\varepsilon_{i\Delta}-\sum_{i=1}^{T}\left(\sum_{k=1}^{K}\phi_kI_{i\Delta}^k\right)^2-2\alpha_0\sum_{i=1}^{T}\sum_{k=1}^{K}\phi_kI_{i\Delta}^k-2e^{\theta\Delta}\sum_{i=1}^{T}y_{(i-1)\Delta}\sum_{k=1}^{K}\phi_kI_{i\Delta}^k-2\lambda_0\sum_{i=1}^{T}\left(\sum_{k=1}^{K}\phi_kI_{i\Delta}^k\right)\varepsilon_{i\Delta}\right].$$

From (a), (b) and (c),

$$\frac{T^{-1/2}}{2e^{\theta\Delta}\lambda_0}\left[\sum_{i=1}^{T}\left(y_{i\Delta}^2-y_{(i-1)\Delta}^2\right)-T\alpha_0^2-\left(e^{2\theta\Delta}-1\right)\sum_{i=1}^{T}y_{(i-1)\Delta}^2-\lambda_0^2\sum_{i=1}^{T}\varepsilon_{i\Delta}^2-2\alpha_0e^{\theta\Delta}\sum_{i=1}^{T}y_{(i-1)\Delta}-2\alpha_0\lambda_0\sum_{i=1}^{T}\varepsilon_{i\Delta}\right]$$
$$\Longrightarrow\frac{\sigma N^{1/2}}{2}\left[\tilde{\Xi}_1^2-\frac{y_0^2}{\sigma^2N}-2c\tilde{\Xi}_3-1+2c\frac{\mu}{\sigma N^{1/2}}\tilde{\Xi}_2\right].$$

Moreover,

$$\sum_{i=1}^{T}\left(\sum_{k=1}^{K}\phi_kI_{i\Delta}^k\right)^2=\sum_{k=1}^{K}\phi_k^2\sum_{i=1}^{T}I_{i\Delta}^k=\sum_{k=1}^{K}\phi_k^2;$$
$$2\alpha_0\sum_{i=1}^{T}\sum_{k=1}^{K}\phi_kI_{i\Delta}^k=2\mu\left(1-e^{\theta\Delta}\right)\sum_{k=1}^{K}\phi_k\sum_{i=1}^{T}I_{i\Delta}^k=2\mu\left(1-e^{\theta\Delta}\right)\sum_{k=1}^{K}\phi_k\to0;$$
$$2\lambda_0\sum_{i=1}^{T}\left(\sum_{k=1}^{K}\phi_kI_{i\Delta}^k\right)\varepsilon_{i\Delta}=2\lambda_0\sum_{k=1}^{K}\phi_k\sum_{i=1}^{T}I_{i\Delta}^k\varepsilon_{i\Delta}=2\lambda_0\sum_{k=1}^{K}\phi_k\varepsilon_{\tau_k\Delta}\to0;$$

and

$$
\begin{aligned}
& 2e^{\theta\Delta}\sum_{i=1}^{T} y_{(i-1)\Delta}\sum_{k=1}^{K}\phi_k I_{i\Delta}^{k} \\
= \ & 2e^{\theta\Delta}\sum_{k=1}^{K}\phi_k\sum_{i=1}^{T} y_{(i-1)\Delta} I_{i\Delta}^{k} = 2e^{\theta\Delta}\sum_{k=1}^{K}\phi_k y_{(\tau_k-1)\Delta} \\
= \ & 2e^{\theta\Delta}\sum_{k=1}^{K}\phi_k\left[\alpha_0\frac{1-e^{(\tau_k-1)\theta\Delta}}{1-e^{\theta\Delta}}+\lambda_0\sum_{j=1}^{\tau_k-1}e^{(\tau_k-1-j)\theta\Delta}\varepsilon_{j\Delta}+e^{(\tau_k-1)\theta\Delta}y_0+\sum_{j=1}^{\tau_k-1}e^{(\tau_k-1-j)\theta\Delta}\sum_{s=1}^{K}\phi_s I_{j\Delta}^{s}\right] \\
\Longrightarrow \ & 2\sigma^2 N\sum_{k=1}^{K}\frac{\phi_k}{N^{1/2}\sigma}\left[\frac{\mu}{N^{1/2}\sigma}\left(1-e^{r_k c}\right)+J_c\left(r_k\right)+e^{r_k c}\frac{y_0}{N^{1/2}\sigma}+\Delta_4\right], \qquad (2.2)
\end{aligned}
$$

where $\Delta_4 = 0$ if $K = 1$ and $\Delta_4 = \sum_{j=1}^{k-1} e^{\left(r_k-r_j\right)\theta}\frac{\phi_j}{N^{1/2}\sigma}$ if $K > 1$. This is because

$$
\alpha_0\frac{1-e^{(\tau_k-1)\theta\Delta}}{1-e^{\theta\Delta}}+\lambda_0\sum_{j=1}^{\tau_k-1}e^{(\tau_k-1-j)\theta\Delta}\varepsilon_{j\Delta}+e^{(\tau_k-1)\theta\Delta}y_0 \Longrightarrow N^{1/2}\sigma\left[\frac{\mu}{N^{1/2}\sigma}\left(1-e^{r_k c}\right)+J_c\left(r_k\right)+e^{r_k c}\frac{y_0}{N^{1/2}\sigma}\right]
$$

and

$$
\sum_{j=1}^{\tau_k-1}e^{(\tau_k-1-j)\theta\Delta}\sum_{s=1}^{K}\phi_s I_{j\Delta}^{s} = \begin{cases} \sum_{s=1}^{K}\phi_s e^{(\tau_k-1)\theta\Delta}\sum_{j=1}^{\tau_k-1}e^{-j\theta\Delta}I_{j\Delta}^{s}=0 & \text{if } K=1 \\ \sum_{j=1}^{k-1}e^{\left(\tau_k-1-\tau_j\right)\theta\Delta}\phi_j \ \to N^{1/2}\sigma\sum_{j=1}^{k-1}e^{\left(r_k-r_j\right)\theta}\frac{\phi_j}{N^{1/2}\sigma} & \text{if } K>1 \end{cases}.
$$

Therefore,

$$
\begin{aligned}
T^{-1/2}\sum_{i=1}^{T}y_{(i-1)\Delta}\varepsilon_{i\Delta} = & \frac{\sigma N^{1/2}}{2}\left\{\tilde{\Xi}_1^2-\frac{y_0^2}{\sigma^2 N}-2c\tilde{\Xi}_3-1+2c\frac{\mu}{\sigma N^{1/2}}\tilde{\Xi}_2-\sum_{k=1}^{K}\frac{\phi_k^2}{\sigma^2 N}\right. \\
& \left.-2\sum_{k=1}^{K}\frac{\phi_k}{N^{1/2}\sigma}\left[\frac{\mu}{N^{1/2}\sigma}\left(1-e^{r_k c}\right)+J_c\left(r_k\right)+e^{r_k c}\frac{y_0}{N^{1/2}\sigma}+\Delta_4\right]\right\} \\
= & \ \sigma N^{1/2}\tilde{\Xi}_4.
\end{aligned}
$$

□

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